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A CONSTRUCTION OF INFINITE SET OF RATIONAL SELF-EQUIVALENCES

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Abstract
In [5] it was shown that two number fields have isomorphic Witt rings of quadratic forms if and only if there is a Hilbert symbol equivalence between them. A *Hilbert symbol equivalence* between two number fields K and L is a pair of maps (t, T), where $t\colon K^*/K^{*2} \to L^*/L^{*2}$ is a group isomorpism and $T\colon \Omega_K \to \Omega_L$ is a bijection between the sets of finite and infinite primes of K and L, respectively, such that the Hilbert symbols are preserved: for any $a, b \in K^*/K^{*2}$ and for any $P \in \Omega_K$,

$$(a, b)_P = (t(a), t(b))_{T(P)}.$$

A Hilbert symbol equivalence between the field $\mathbb{Q}$ and itself is called *rational self-equivalence*. In [5] the authors present a construction of equivalence of two fields starting from the so called Hilbert small equivalence of two fields. We use this idea for constructing infinite set of rational self-equivalences.

1. Introduction

Consider the field $\mathbb{Q}$ of all rational numbers. Let $\mathbb{P}$ denote the set of all prime numbers together with the symbol ∞. For every prime number, a complement $\mathbb{Q}_p$ of the field $\mathbb{Q}$ is defined with the help of valuation v_p called a *p-adic number field*. Moreover, we agree that $\mathbb{Q}_\infty = \mathbb{R}$ is a complement of the field $\mathbb{Q}$ with respect to the usual absolute value (cf. [1]).

Definition 1.1. Let t be an automorphism of the group of square classes

$$t\colon \mathbb{Q}^*/\mathbb{Q}^{*2} \to \mathbb{Q}^*/\mathbb{Q}^{*2}$$

and T be a bijection of the form

$$T\colon \mathbb{P} \to \mathbb{P}$$

preserving Hilbert symbols in the sense that

$$(a,b)_p = (t(a), t(b))_{T(p)}$$

for all $a, b \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ and all $p \in \mathbb{P}$. The pair (t, T) is said to be a *Hilbert symbol rational self-equivalence.*

The notion of rational self-equivalence is a special case of Hilbert symbol equivalence of fields, where the prime numbers are replaced by prime ideals of global fields (cf. [4], [5]).

From [5] we know that two number fields have isomorphic Witt rings if there exists *Hilbert symbol equivalence* (see the above definition) beetwen them, called in [5] *reciprocity equivalence of fields* (cf. [5], Theorem 1). It was shown in [5] that the bijection t defined above induces the strong isomorphism of Witt rings of those fields (more about the isomorphisms of Witt rings the reader can find in [7]). The authors have presented a construction of Hilbert symbol equivalence of fields starting with the so-called *small equivalence*. We shall use this idea for constructing rational self-equivalences. We recall some notions and facts (cf. [3], [4], [5]) which are used in the next part of the present paper.

A finite, nonempty set $S \subset \mathbb{P}$ containing 2 and ∞ is called *sufficiently large.*

Let S be a sufficiently large set of prime numbers $S = \{p_1, \ldots, p_n\}$ and assume that $p_1 = \infty$, $p_2 = 2$. We define the *set of S-singular elements* as follows:

$$E_S = \{x \in \mathbb{Q}^* \ : \ v_p(x) \equiv 0 \pmod 2 \ \text{ for all } \ p \notin S\}.$$

Notice that E_S is a subgroup of the multiplicative group of the field $\mathbb{Q}$ containing all squares of rational numbers. Therefore, the quotient group $E_S/\mathbb{Q}^{*2}$ is a subgroup of the group $\mathbb{Q}^*/\mathbb{Q}^{*2}$.

By the definition of the set E_S, we get that every element $x \in \mathbb{Q}$ has the following factorization

$$x = (-1)^{e_1} 2^{2k_2+e_2} p_3^{2k_3+e_3} \cdots p_n^{2k_n+e_n} q_1^{2l_1} \cdots q_m^{2l_m},$$

where $q_1, q_2, \ldots, q_m \notin S$ are prime numbers, $k_i, l_i \in \mathbb{Z}$ and $e_i \in \{0, 1\}$. Then

$$x\mathbb{Q}^{*2} = (-1)^{e_1} 2^{e_2} p_3^{e_3} \cdots p_n^{e_n} \mathbb{Q}^{*2}.$$

It follows that the elements of the group $E_S/\mathbb{Q}^{*2}$ are represented by the integers of the form $(-1)^{e_1}2^{e_2}p_3^{e_3}\cdots p_n^{e_n}$ in the unique way.

For every $p \in \mathbb{P}$, the natural imbedding of the field $\mathbb{Q}$ in the field $\mathbb{Q}_p$ induces the group homomorphism $i_p\colon \mathbb{Q}^*/\mathbb{Q}^{*2} \to \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ which is surjective. For the finite set $S = \{p_1, \dots, p_n\} \subset \mathbb{P}$ we get the dual homomorphism

$$\mathrm{diag}_S : \mathbb{Q}^*/\mathbb{Q}^{*2} \longrightarrow \prod_{p\in S} \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$$

defined by

$$\mathrm{diag}_S(a) = [i_{p_1}(a), \dots, i_{p_n}(a)] = [a\mathbb{Q}_{p_1}^{*2}, \dots, a\mathbb{Q}_{p_n}^{*2}].$$

Let us introduce the following notation $G_S := \prod_{p\in S} \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$. The restriction of the homomorphism diag to the set $E_S/\mathbb{Q}^{*2}$ is denoted by i_S.

Definition 1.2. Let S be sufficiently large set of prime numbers defined as above. A *small S-equivalence* is a pair $\mathcal{R} = ((t_p)_{p\in S}, T)$, where

1) $T\colon S \to T(S)$ is a bijection,

2) there exists the isomorphism of the groups of square classes $t_S\colon E_S/\mathbb{Q}^{*2} \to E_{T(S)}/\mathbb{Q}^{*2}$,

3) $(t_p)_{p\in S}$ is a family of local isomorphisms $t_p : \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \to \mathbb{Q}_{T(p)}^*/\mathbb{Q}_{T(p)}^{*2}$ preserving Hilbert symbols, i.e.

$$(a, b)_p = (t_p(a), t_p(b))_{T(p)} \quad \text{for all} \quad a, b \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2},$$

4) the following diagram commutes

$$\begin{array}{ccc} E_S/\mathbb{Q}^{*2} & \xrightarrow{i_S} & \prod_{p\in S} \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \\ \downarrow t_S & & \downarrow \prod t_p \\ E_{T(S)}/\mathbb{Q}^{*2} & \xrightarrow{i_{T(S)}} & \prod_{p\in S} \mathbb{Q}_{T(p)}^*/\mathbb{Q}_{T(p)}^{*2} \end{array}$$

We shall show that the small S-equivalence defined on arbitrary sufficiently large set S can be extended to rational self-equivalence. Next we will present how to change the construction in order to get the infinite set of rational self-equivalences.

2. The construction of rational self-equivalence

Let $S = S_2 = \{\infty, 2\}$. It is easy to construct the natural small S_2-equivalence. It suffices to define the map T by $T(p) = p$ for both $p \in S_2$, so we have $T(S_2) = S_2$ and we define the map t_p as follows: $t_\infty\colon \mathbb{R}^*/\mathbb{R}^{*2} \to \mathbb{R}^*/\mathbb{R}^{*2}$ as identity and $t_2\colon \mathbb{Q}_2^*/\mathbb{Q}_2^{*2} \to \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ as identity too. Then the pair $((t_p)_{p\in S_2}, T)$ is a natural small S_2-equivalence, and suitable conditions of the definition are fulfilled in the obvious way.

Lemma 2.1. *Let p_3, q_3 be a prime numbers outside of S_2 and assume that $S_3 := S_2 \cup \{p_3\} = \{\infty, 2, p_3\}$ and $T(S_3) := S_2 \cup \{q_3\} = \{\infty, 2, q_3\}$. If $p_3 \equiv q_3 (\mathrm{mod}\ 8)$, then there exists a small S_3-equivalence.*

P r o o f. It is obvious that S_3 and $T(S_3)$ are sufficiently large sets.
We define the bijection T in the following way:

$$T(\infty) = \infty, \quad T(2) = 2 \text{ and } T(p_3) = q_3.$$

Now we define the maps t_p for all $p \in S_3$. We do not change the maps t_∞ and t_2, that is let $t_\infty\colon \mathbb{R}^*/\mathbb{R}^{*2} \to \mathbb{R}^*/\mathbb{R}^{*2}$ be identity and $t_2\colon \mathbb{Q}_2^*/\mathbb{Q}_2^{*2} \to \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ be identity too. Next, let $t_{p_3}\colon \mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2} \to \mathbb{Q}_{q_3}^*/\mathbb{Q}_{q_3}^{*2}$ be an isomorphism from the group $\mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2} := \{1, p_3, u, up_3\}$ to the group $\mathbb{Q}_{q_3}^*/\mathbb{Q}_{q_3}^{*2} := \{1, q_3, v, vq_3\}$ defined by

$$t_{p_3}(1) = 1, \quad t_{p_3}(p_3) = q_3, \quad t_{p_3}(u) = v, \quad t_{p_3}(up_3) = vq_3.$$

The maps t_∞ and t_2 are identities, so they preserve Hilbert symbols in the obvious way. We shall show that t_{p_3} preserves the Hilbert symbol too.

As we know, the Legendre symbol $\left(\frac{u}{p_3}\right) = -1$ in the group $\mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2}$. By the properties of the Hilbert symbol (cf. [6], Theorem 3, p. 23), we get:

1) $(a, 1)_{p_3} = 1$ for every $a \in \mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2}$

2) $(p_3, p_3)_{p_3} = (-1, p_3)_{p_3} = \left(\frac{-1}{p_3}\right)$

3) $(u, u)_{p_3} = (-1, u)_{p_3} = 1$

4) $(p_3, u)_{p_3} = \left(\frac{u}{p_3}\right) = -1$

5) $(p_3, up_3)_{p_3} = (p_3, u)_{p_3} \cdot (p_3, p_3)_{p_3} = -1 \cdot (-1, p_3)_{p_3} = -\left(\frac{-1}{p_3}\right)$

6) $(u, up_3)_{p_3} = (u, u)_{p_3} \cdot (u, p_3)_{p_3} = 1 \cdot (-1) = -1$

7) $(up_3, up_3)_{p_3} = (u, up_3)_{p_3} \cdot (p_3, up_3)_{p_3} = -1 \cdot \left(-\left(\frac{-1}{p_3}\right)\right) = \left(\frac{-1}{p_3}\right)$

The above Hilbert symbols depend only on Legendre symbols, so from the condition $p_3 \equiv q_3 (\text{mod } 8)$ it follows that we get the same values of Hilbert symbols $(\ , \ )_{q_3}$ defined with the map t_{p_3} in the group $\mathbb{Q}^*_{q_3}/\mathbb{Q}^{*2}_{q_3}$.

Moreover, if $a, b \in \mathbb{Q}^*$ are in the same square class in the group $\mathbb{Q}^*_{p_3}/\mathbb{Q}^{*2}_{p_3}$, then their images by the map t_{p_3} belong to the same square class in the group $\mathbb{Q}^*_{q_3}/\mathbb{Q}^{*2}_{q_3}$ and the Hilbert symbols are preserved in a similar way as above. Hence, the maps t_{p_i} preserve Hilbert symbols for all $p_i \in S_3$, $i = 1, 2, 3$.

Now we are going to show the commutativity of the diagram.
Notice that in our case the both groups $E_{S_3}/\mathbb{Q}^{*2}$ and $E_{T(S_3)}/\mathbb{Q}^{*2}$ have 8 elements. It suffices to show the commutativity of the following diagram on the generators of the group $E_{S_3}/\mathbb{Q}^{*2}$:

$$\begin{array}{ccc} E_{S_3}/\mathbb{Q}^{*2} & \xrightarrow{i_{S_3}} & \mathbb{R}^*/\mathbb{R}^{*2} \times \mathbb{Q}^*_2/\mathbb{Q}^{*2}_2 \times \mathbb{Q}^*_{p_3}/\mathbb{Q}^{*2}_{p_3} \\ \downarrow t_{S_3} & & \downarrow \prod t_{p_i} \\ E_{T(S_3)}/\mathbb{Q}^{*2} & \xrightarrow{i_{T(S_3)}} & \mathbb{R}^*/\mathbb{R}^{*2} \times \mathbb{Q}^*_2/\mathbb{Q}^{*2}_2 \times \mathbb{Q}^*_{q_3}/\mathbb{Q}^{*2}_{q_3} \end{array}$$

1. It is obvious that for $a = 1 \in E_{S_3}/\mathbb{Q}^{*2}$ the diagram commutes.

2. Analogously, the diagram commutes for $a = -1$.
In fact, going from $E_{S_3}/\mathbb{Q}^{*2}$ to the right, we get

$$(\prod t_{p_i} \circ \ i_{S_3})(-1) = \prod t_{p_i}(i_{S_3}(-1)) = \prod t_{p_i}(i_\infty(-1), i_2(-1), i_{p_3}(-1)) =$$
$$= \prod t_{p_i}(-1, -1, -1) = (-1, -1, -1)$$

and, on the other hand, going down, we get

$$(i_{T(S_3)} \circ t_{S_3})(-1) = i_{T(S_3)}(t_{S_3}(-1)) = i_{T(S_3)}(t_{S_3}((-1)^1 \cdot 2^0 \cdot p_3^0)) =$$
$$= i_{T(S_3)}((-1)^1 \cdot 2^0 \cdot q_3^0) = i_{T(S_3)}(-1) =$$
$$= (i_\infty(-1), i_2(-1), i_{q_3}(-1)) = (-1, -1, -1).$$

It remains to check commutativity on 2 generators of the group $E_{S_3}/\mathbb{Q}^{*2}$.

3. Let $a = 2$. Then

$$(\prod t_{p_i} \circ \ i_{S_3})(a) = (\prod t_{p_i} \circ \ i_{S_3})(2) = \prod t_{p_i}(i_{S_3}(2)) =$$
$$= \prod t_{p_i}(i_\infty(2), i_2(2), i_{p_3}(2)) = = \prod t_{p_i}(2, 2, 2) = (1, 2, t_{p_3}(2))$$

and, on the other hand,

$$(i_{T(S_3)} \circ t_{S_3})(2) = i_{T(S_3)}(t_{S_3}(2)) = i_{T(S_3)}(t_{S_3}((-1)^0 \cdot 2^1 \cdot p_3^0)) =$$
$$= i_{T(S_3)}((-1)^0 \cdot 2^1 \cdot q_3^0) = i_{T(S_3)}(2) = (i_\infty(2), i_2(2), i_{q_3}(2)) = (1, 2, 2).$$

Since $p_3 \equiv q_3 (\text{mod } 8)$, then $\left(\frac{2}{p_3}\right) = \left(\frac{2}{q_3}\right)$. Therefore, $2 \in \{\mathbb{Q}_{p_3}^{*2}, u\mathbb{Q}_{p_3}^{*2}\}$, hence 2 is a square in $\mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2}$ if and only if it is a square in $\mathbb{Q}_{q_3}^*/\mathbb{Q}_{q_3}^{*2}$, hence if $2 \in u\mathbb{Q}_{p_3}^{*2}$, then $t_{p_3}(2) \in v\mathbb{Q}_{q_3}^{*2}$. It follows that $t_{p_3}(2) = 2$, what we need in this case.

4. Now let $a = p_3$. Then

$$(\prod t_{p_i} \circ\ i_{S_3})(a) = (\prod t_{p_i} \circ\ i_{S_3})(p_3) = \prod t_{p_i}(i_{S_3})(p_3)) =$$
$$= \prod t_{p_i}(i_\infty(p_3), i_2(p_3), ip_3(p_3)) = = \prod t_{p_i}(1, p_3, p_3) = (1, t_2(p_3), q_3)$$

and, on the other hand, we get

$$(i_{T(S_3)} \circ t_{S_3})(p_3) = i_{T(S_3)}(t_{S_3}(p_3)) = i_{T(S_3)}(t_{S_3}((-1)^0 \cdot 2^0 \cdot p_3^1)) =$$
$$= i_{T(S_3)}((-1)^0 \cdot 2^0 \cdot q_3^1) = i_{T(S_3)}(q_3) =$$
$$= (i_\infty(q_3), i_2(q_3), i_{q_3}(q_3)) = (1, q_3, q_3).$$

Observe that the condition $p_3 \equiv q_3 \,(\text{mod } 8)$ implies $p_3 q_3 \equiv 1 \,(\text{mod } 8)$, because in the ring $\mathbb{Z}/8\mathbb{Z}$ every square of unit element equals to 1. By Hensel's Lemma (cf. [2], Appendix C, p. 83) it follows that $p_3 q_3 \in \mathbb{Q}_2^{*2}$, hence p_3 and q_3 belong to the same square class in $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$. This means that $t_2(p_3) = q_3$, and we get commutativity in this case too.

The above steps 1.-4. fulfill requirements for commutativity of considered diagram for the elements of the group $E_{S_3}/\mathbb{Q}^{*2}$. That finishes the proof of existing required small S_3-equivalence.

Remark 2.2. Notice that the condition $p_3 \equiv q_3 \,(\text{mod } 4)$ is not sufficient. In fact, let $p_3 = 5$ and $q_3 = 17$. Then $p_3 \equiv q_3 \,(\text{mod } 4)$, but $p_3 \not\equiv q_3 \,(\text{mod } 8)$ and $-1 = \left(\frac{2}{5}\right) \neq \left(\frac{2}{17}\right) = 1$, hence in this case we will not get commutativity of the diagram, because we have $2 \notin \mathbb{Q}_{p_3}^{*2}$ and $2 \in \mathbb{Q}_{q_3}^{*2}$.

Lemma 2.3. *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{-1, 1\}$ and let $q_1, q_2, \dots, q_k$, $k \in \mathbb{N}$ be the prime numbers. There exist infinite set of prime numbers q such that*

$$\left(\frac{q}{q_i}\right) = \varepsilon_i, \qquad 1 \le i \le k.$$

P r o o f. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{-1, 1\}$ and let $q_1, q_2, \dots, q_k$ be arbitrary prime numbers. One can choose values a_i such that $1 \leq a_i \leq q_i$ and $\left(\frac{a_i}{q_i}\right) = \varepsilon_i$ (such a_i exist). By the Chinese remainder theorem, there exists $a \in \{1, 2, \dots, m\}$, where $m = q_1 \cdot \ldots \cdot q_k$ such that $a \equiv a_i (\mathrm{mod}\ q_i)$. Then

$$\left(\frac{a}{q_i}\right) = \varepsilon_i = \left(\frac{a_i}{q_i}\right),$$

since Legendre symbols depend on the residue from dividing by q_i.
We construct an atrithmetic sequence

$$t_l = a + l \cdot m, \qquad l = 1, 2, \dots, \infty. \tag{1}$$

One could use Dirichlet's theorem provided that $(a, m) = 1$. Assume that $(a, m) = b \neq 1$, i.e. there exists b such that $b|a$ and $b|m$. Then $q_i|b$ for some i and $a \equiv 0(\mathrm{mod}\ q_i)$ which contradicts $a_i \not\equiv 0(\mathrm{mod}\ q_i)$. So we have $(a, m) = 1$ and we can use Dirichlet's theorem. Therefore, in the sequence (1) there exist infinitely many prime numbers such that

$$\left(\frac{t_l}{q_i}\right) = \varepsilon_i.$$

Lemma 2.4. *Let $S_k = \{\infty, 2, p_3, p_4, \dots, p_k\}$, $S'_k = \{\infty, 2, q_3, q_4, \dots, q_k\}$, $k \in \mathbb{N}$ be two sufficiently large sets of prime numbers and assume that $\mathcal{R}_{S_k} = ((t_p)_{p \in S_k}, T)$ is a small S_k-equivalence. Let q_{k+1} be the smallest prime number with $q_{k+1} \in \mathbb{P} \setminus S'_k$ and denote the set $S'_{k+1} \colon = S'_k \cup \{q_{k+1}\}$. Then the small S_k-equivalence $\mathcal{R}_{S_k}$ can be extended to a small S'_{k+1}-equivalence.*

P r o o f. Let $S_k = \{\infty, 2, p_3, p_4, \dots, p_k\}$ and $S'_k = \{\infty, 2, q_3, q_4, \dots, q_k\}$ be two sufficiently large sets and assume that we get the small S_k-equivalence with the maps $((t_p)_{p \in S_k}, T)$ by constructions of extending small equivalences defined on the sets S_i and S'_i, $3 \leq i \leq k$.

Choose the smallest prime number $p_{k+1} \in \mathbb{P} \setminus S_k$ which fulfills the following conditions:

1) $p_{k+1} \equiv q_{k+1}\ (\mathrm{mod}\ 8)$,
2) $\left(\frac{p_i}{p_{k+1}}\right) = \left(\frac{q_i}{q_{k+1}}\right)$ for all $3 \leq i \leq k$

(by Lemma 2.3 there exist infinitely many such prime numbers).
Denote $S_{k+1} \colon = S_k \cup \{p_{k+1}\}$. We define maps

$$t_{q_i} \colon \mathbb{Q}^*_{q_i}/\mathbb{Q}^{*2}_{q_i} \to \mathbb{Q}^*_{p_i}/\mathbb{Q}^{*2}_{p_i}, \quad 1 \leq i \leq k$$

and
$$T' = T^{-1}\colon S'_k \to S_k$$
as the inverse of the maps t_{p_i} and T.
Next we set $T'(q_{k+1}) = p_{k+1}$ and $t_{q_{k+1}}(q_{k+1}) = p_{k+1}$, $t_{q_{k+1}}(u_{k+1}) = v_{k+1}$. We will show that the pair $(t_{q_{k+1}}, T') = (t^{-1}_{p_{k+1}}, T^{-1})$ is the required small S'_{k+1}-equivalence.

The condition $p_{k+1} \equiv q_{k+1} \pmod 8$ guarantees the preserving of the Hilbert symbols in the prime number q_{k+1} by analogous calculations as in Lemma 2.1.

We will check now the commutativity of the suitable diagram. The group $E_{S_{k+1}}/\mathbb{Q}^{*2}$ has 2^{k+1} elements. In this case we have
$$G_{S_{k+1}} = \mathbb{R}^*/\mathbb{R}^{*2} \times \mathbb{Q}_2^*/\mathbb{Q}_2^{*2} \times \mathbb{Q}_{p_3}^*/\mathbb{Q}_{p_3}^{*2} \times \cdots \times \mathbb{Q}_{p_k}^*/\mathbb{Q}_{p_k}^{*2} \times \mathbb{Q}_{p_{k+1}}^*/\mathbb{Q}_{p_{k+1}}^{*2},$$
$$G_{S'_{k+1}} = \mathbb{R}^*/\mathbb{R}^{*2} \times \mathbb{Q}_2^*/\mathbb{Q}_2^{*2} \times \mathbb{Q}_{q_3}^*/\mathbb{Q}_{q_3}^{*2} \times \cdots \times \mathbb{Q}_{q_k}^*/\mathbb{Q}_{q_k}^{*2} \times \mathbb{Q}_{q_{k+1}}^*/\mathbb{Q}_{q_{k+1}}^{*2},$$
and the corresponding diagram has the following form
$$\begin{array}{ccc} E_{S'_{k+1}}/\mathbb{Q}^{*2} & \overset{i_{S'_{k+1}}}{\longrightarrow} & G_{S'_{k+1}} \\ \downarrow t_{S'_{k+1}} & & \downarrow \prod t_{q_i} \\ E_{S_{k+1}}/\mathbb{Q}^{*2} & \overset{i_{S_{k+1}}}{\longrightarrow} & G_{S_{k+1}} \end{array}$$
We will consider the commutativity on the generators of the group $E_{S'_{k+1}}$.

1. Let $a = 2$. Then
$$\prod t_{q_i}(i_{S'_{k+1}}(a)) = \prod t_{q_i}(i_{S'_{k+1}}(2)) = \prod t_{q_i}(2, \dots, 2) =$$
$$= (1, 2, t_{q_3}(2), \dots, t_{q_k}(2), t_{q_{k+1}}(2)).$$
On the other hand, we get
$$i_{S_{k+1}}(t_{S'_{k+1}}(2)) = i_{S_{k+1}}(t_{S'_{k+1}}((-1)^0 \cdot 2^1 \cdot q_3^0 \cdot \ldots \cdot q_k^0 \cdot q_{k+1}^0)) =$$
$$= i_{S_{k+1}}((-1)^0 \cdot 2^1 \cdot p_3^0 \cdot \ldots \cdot p_k^0 \cdot p_{k+1}^0) = i_{S_{k+1}}(2) = (1, 2, \dots, 2)\ .$$
Now we need to show that $t_{p_i}(2) = 2$ for $3 \leq i \leq k+1$. By $p_i \equiv q_i \pmod 8$ and from the assumption of the way of extensions construction, we get the above result for $1 \leq i \leq k$. Since we have chosen the prime number p_{k+1} fulfilling the condition $p_{k+1} \equiv q_{k+1} \pmod 8$, then we get the result for $i = k+1$.

2. Consider $a = q_s$ for any $3 \le s \le k$. Then:

$$\prod t_{q_i}(i_{S'_{k+1}}(q_s)) = \prod t_{q_i}(i_{S'_{k+1}}(q_s)) = \prod t_{q_i}(1, q_s, \dots, q_s) =$$
$$= (1, t_2(q_s), \dots, p_s, \dots, t_k(q_s), t_{k+1}(q_s))$$

and, on the other hand,

$$i_{S_{k+1}}(t_{S'_{k+1}}(q_s)) = i_{S_{k+1}}(t_{S'_{k+1}}((-1)^0 \cdot 2^0 \cdot q_3^0 \cdot \ldots \cdot q_s^1 \cdot \ldots \cdot q_k^0 \cdot q_{k+1}^0)) =$$
$$= i_{S_{k+1}}((-1)^0 \cdot 2^0 \cdot p_3^0 \cdot \ldots \cdot p_s^1 \cdot \ldots \cdot p_k^0 \cdot p_{k+1}^0) = i_{S_{k+1}}(p_s) = (1, p_s, \dots, p_s).$$

Hence, we have to show that $t_i(q_s) = p_s$ for all $3 \le i \le k+1$. Of course, it is true for $3 \le i \le k$ by induction hypothesis. Consider $i = k+1$. Since we have $\left(\frac{p_s}{p_{k+1}}\right) = \left(\frac{q_s}{q_{k+1}}\right)$ and $p_s \in \{1, v_{k+1}\} \in \mathbb{Q}^*_{p_{k+1}}/\mathbb{Q}^{*2}_{p_{k+1}}$, then we get $t_{k+1}(q_s) = p_s$.

3. It remains to check the element $a = q_{k+1}$. We have:

$$\prod t_{q_i}(i_{S'_{k+1}}(q_{k+1})) = \prod t_{q_i}(1, q_{k+1}, \dots, q_{k+1}) =$$
$$= (1, t_2(q_{k+1}), \dots, t_k(q_{k+1}), p_{k+1})$$

and, on the other hand,

$$i_{S_{k+1}}(t_{S'_{k+1}}(q_{k+1})) = i_{S_{k+1}}(t_{S'_{k+1}}((-1)^0 \cdot 2^0 \cdot q_3^0 \cdot \ldots \cdot q_k^0 \cdot q_{k+1}^1)) =$$
$$= i_{S_{k+1}}((-1)^0 \cdot 2^0 \cdot p_3^0 \cdot \ldots \cdot p_k^0 \cdot p_{k+1}^1) = i_{S_k}(p_{k+1}) = (1, p_{k+1}, \dots, p_{k+1}).$$

Now by $p_{k+1} \equiv q_{k+1} (\mathrm{mod}\ 8)$, it follows that $t_2(q_{k+1}) = p_{k+1}$ and similarly to the previous step of the proof, by $\left(\frac{p_{k+1}}{p_i}\right) = \left(\frac{q_{k+1}}{q_i}\right)$ for $3 \le i \le k+1$, it follows that $t_i(q_{k+1}) = p_{k+1}$ for $3 \le i \le k+1$.

The diagram is commutative, what finishes the proof.

Theorem 2.5. *The small S_3-equivalence in Lemma 2.1 can be extended to the rational self-equivalence.*

P r o o f. We use induction on k, where k is the index of the following prime numbers in the natural order.

Let $k = 2$. By Lemma 2.1, the natural small S_2-equivalence can be extended to a small S_3-equivalence.

Now add a prime number outside of S_3 and extend the obtained S_3-equivalence to some S'_4-equivalence with the help of analogous construction, where $S'_3 \subseteq S'_4$. Then add another prime to the obtained set S_4 and extend the small equivalence. We can imagine the remaining prime numbers written in two sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ in natural order. Let us continue the procedure described above to extend the equivalences by adding the smallest prime number p_i outside of S_{i-1}, $(i > 1)$ to the set S_i and choosing q_i for extending the small S_i-equivalence, and then taking the smallest prime number q_{i+1} outside the set S'_i and choosing a prime number p_{i+1}, for extending to a small S'_{i+1}-equivalence.

In this way, we use all prime numbers in both sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$. By Lemma 2.3, it follows that there exist infinitely many prime numbers which fulfill the required conditions involving Legendre symbol, so in every step of the construction described above the suitable diagram can be made commutative and the small equivalence can be extended.

Suppose that we have a small S_k-equivalence for some odd prime number k obtained by the construction described above. Then a small S_k-equivalence can be extended to a small S'_{k+1}-equivalence by Lemma 2.4.

Continue the process of extending small equivalences for infinite sequences of prime numbers $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$. By the following constructions, we obtain the sets S_i, S'_i such that $S_2 \subseteq S_i, S'_i$, the bijection

$$T\colon \mathbb{P} \to \mathbb{P}$$

and the group isomorphisms

$$t_{p_i}\colon \mathbb{Q}_{p_i}^*/\mathbb{Q}_{p_i}^{*2} \to \mathbb{Q}_{q_i}^*/\mathbb{Q}_{q_i}^{*2}$$

preserving Hilbert symbols as follows

$$(x,y)_{p_i} = (t_{p_i}(x), t_{p_i}(y))_{q_i}$$

for all $x, y \in \mathbb{Q}_{p_i}^*$ and all $p_i \in \mathbb{P} \setminus S_i$.

Since any square-class x in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ lies in the group $E_{S_k}/\mathbb{Q}^{*2}$ for some finite set S_k containing S_2, we can define an automorphism t of the group $\mathbb{Q}^*/\mathbb{Q}^{*2}$ by setting

$$t(x) = t_{S_k}(x).$$

The above construction assures us that for arbitrary $i < j$ an isomorphism t_{S_j} is an extension of the isomorphism t_{S_i}, hence the map t defined above does not depend on the choice of the set S_k.

Therefore, the above construction is an extension of the small S_3-equivalence defined in Lemma 2.1 to some rational self-equivalence (t, T), as required.

Theorem 2.6. *Let $S \subseteq \mathbb{P}$ be a sufficiently large set. Any small S-equivalence can be extended to a rational self-equivalence.*

Compare the proof of [5], Theorem 2.

Corollary 2.7. *Let* (t, T) *be a rational self-equivalence constructed as in the proof of theorem 2.5. Then the map* $t\colon \mathbb{Q}^*/\mathbb{Q}^{*2} \to \mathbb{Q}^*/\mathbb{Q}^{*2}$ *induces a strong automorphism in a Witt ring* $W(\mathbb{Q})$ *of the field of rational numbers.*

P r o o f follows from [5], Corollary 1.

3. Alternative constructions of rational self-equivalences

In this section, we shall show how to ramify the construction of rational self-equivalence described in the previous section in order to get other rational self-equivalences. By changing the choice of prime numbers, we will obtain infinitely many rational self-equivalences. We shall present some examples showing the first steps of such constructions made by complex computer programme.

Let $(\mathbb{P}, <)$ denote the set of all prime numbers together with the symbol ∞ ordered in a natural way by the relation $<$ and under assumption that $p_1 = \infty$, $p_2 = 2$. The construction described in the proof of theorem 2.5 is based on choosing the smallest prime number which lies outside of the given sufficiently large set and fulfills some required conditions. However, we can consider many permutations of the set $\mathbb{P}$ of all prime numbers, that means we can consider the ordered set $(\mathbb{P}, \alpha)$ instead of $(\mathbb{P}, <)$, where α denotes another order of prime numbers in $\mathbb{P}$. Of course, there exist infinitely many such permutations α.

One can suppose that any permutation α of the set $\mathbb{P}$ of all prime numbers leads to another way of choosing prime numbers and consequently gives another rational self-equivalence. Below we shall show that this is not true.

We shall start with the case, where the procedure coincides with the one described in theorem 2.5, so the following prime numbers are chosen from the natural ordered set $(\mathbb{P}, <)$. Then we obtain the following sequence of substitutions.

Let $p_1 = q_1 = \infty, p_2 = q_2 = 2$. The first, smallest prime number outside of the set S_2 is $p_3 = 3$. According to the described procedure, we choose a prime number q_3 which has the required properties for extension of small S_2-equivalence to small S_3-equivalence as it was described in Lemma 2.1. It turns out to be prime number 11. Let us denote this step of construction in the following way:

1) $p_3 = 3 \to 11 = q_3$.

(Notice that we have to assume that $p_3 \neq q_3$. If we take $p_3 = q_3$, we get an identity).

Next, we take the smallest prime number q_4 which was not used in a sequence $\{q_i\}_{i=1}^{\infty}$. It is number 3. We choose for $q_4 = 3$ the smallest prime number p_4 which has required properties. It is the number 19. Hence, we denote the second step of the construction:
2) $p_4 = 19 \leftarrow 3 = q_4$.
Further constructions of small equivalences can be denoted as follows:

$$
\begin{array}{rlcr}
3) & p_5 = 5 & \rightarrow & 13 = q_5 \\
4) & p_6 = 13 & \leftarrow & 5 = q_6 \\
5) & p_7 = 7 & \rightarrow & 223 = q_7 \\
6) & p_8 = 1103 & \leftarrow & 7 = q_8 \\
7) & p_9 = 11 & \rightarrow & 283 = q_9 \\
8) & p_{10} = 6329 & \leftarrow & 17 = q_{10} \\
9) & p_{11} = 17 & \rightarrow & 2689 = q_{11} \\
10) & p_{12} = 347 & \leftarrow & 19 = q_{12} \\
11) & p_{13} = 23 & \rightarrow & 31159 = q_{13} \\
12) & p_{14} = 77551 & \leftarrow & 23 = q_{14} \\
13) & p_{15} = 29 & \rightarrow & 109229 = q_{15} \\
14) & p_{16} = 138581 & \leftarrow & 29 = q_{16} \\
15) & p_{17} = 31 & \rightarrow & 1010903 = q_{17} \quad \text{etc.}
\end{array}
$$

Therefore we get the following form of the map $T\colon \mathbb{P} \rightarrow \mathbb{P}$:

$$
\begin{aligned}
&T(\infty) = \infty,\\
&T(2) = 2,\\
&T(3) = 11,\\
&T(19) = 3,\\
&T(5) = 13,\\
&T(13) = 5,\\
&T(7) = 223,\\
&T(1103) = 7,\\
&T(11) = 283,\\
&T(6329) = 17,\\
&T(17) = 2689,\\
&T(347) = 19,\\
&T(23) = 31159,\\
&T(77551) = 23,\\
&T(29) = 109229,\\
&T(138581) = 29,\\
&T(31) = 1010903, \quad \text{etc.}
\end{aligned}
$$

It turns out that not every permutation α of the set $\mathbb{P}$ of all prime numbers gives another rational self-equivalence. Assume that the prime numbers in a sequence $\{p_i\}_{i=1}^{\infty}$ are chosen from the set $(\mathbb{P}, <)$ ordered in a natuaral way by the usual relation $<$, as before. Let us now choose the prime numbers in a sequence $\{q_i\}_{i=1}^{\infty}$ from the set $(\mathbb{P}, \alpha)$, where α is a permutation of the set $\mathbb{P}$ such that the only difference between the considered case and $(\mathbb{P}, <)$ is changing the places of the numbers 11 and 13.

Then the following searched prime numbers create the same sequences as in the natural order of prime numbers, so we obtain the same rational self-equivalence as before. The replacement of the numbers 11 and 13 does not change the maps t and T which establish the rational self-equivalence. This proves that not every permutation of the set $\mathbb{P}$ gives another rational self-equivalence.

Now we shall show how to construct infinitely many rational self-equivalences. Notice that according to Lemma 2.3, there exist infinitely many prime numbers which can be chosen in step 1) for the number q_3. The sufficiently condition is $p_3 \equiv q_3 \pmod 8$. If we take another prime number as q_3, we get another maps t and T and consequently another rational self-equivalence. In fact, the map T is different if we take different numbers q_3, namely the difference is the value $T(3)$. Similarly, by the definition of the map t in theorem 2.6, it follows that different elements q_3 give different map t. Thus, in such a way we can construct infinitely many rational self-equivalences.

According to the described construction, searching prime numbers continues for all (infinitely many) prime numbers.

Example 1.

1)	$p_3 = 3$	$\rightarrow$	$19 = q_3$,	then	$T(3)$	$=$	19
2)	$p_4 = 11$	$\leftarrow$	$3 = q_4$,	then	$T(11)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$13 = q_5$,	then	$T(5)$	$=$	13
4)	$p_6 = 13$	$\leftarrow$	$5 = q_6$,	then	$T(13)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$1103 = q_5$,	then	$T(7)$	$=$	1103
6)	$p_8 = 223$	$\leftarrow$	$7 = q_8$,	then	$T(223)$	$=$	7
7)	$p_9 = 17$	$\rightarrow$	$281 = q_9$,	then	$T(17)$	$=$	281
8)	$p_{10} = 8707$	$\leftarrow$	$11 = q_{10}$,	then	$T(8707)$	$=$	11
9)	$p_{11} = 19$	$\rightarrow$	$347 = q_{11}$,	then	$T(19)$	$=$	347
10)	$p_{12} = 4201$	$\leftarrow$	$17 = q_{12}$,	then	$T(4201)$	$=$	17
11)	$p_{13} = 23$	$\rightarrow$	$77551 = q_{13}$,	then	$T(23)$	$=$	77551
12)	$p_{14} = 26119$	$\leftarrow$	$23 = q_{14}$,	then	$T(26119)$	$=$	23
13)	$p_{15} = 29$	$\rightarrow$	$9461 = q_{15}$,	then	$T(29)$	$=$	9461
14)	$p_{16} = 228461$	$\leftarrow$	$29 = q_{16}$,	then	$T(228461)$	$=$	29
15)	$p_{17} = 31$	$\rightarrow$	$3498823 = q_{17}$,	then	$T(31)$	$=$	3498823

etc.

Example 2.

1)	$p_3 = 3$	$\rightarrow$	$43 = q_3$,	then	$T(3)$	$=$	43
2)	$p_4 = 11$	$\leftarrow$	$3 = q_4$,	then	$T(11)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$157 = q_5$,	then	$T(5)$	$=$	157
4)	$p_6 = 173$	$\leftarrow$	$5 = q_6$,	then	$T(173)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$23 = q_5$,	then	$T(7)$	$=$	23
6)	$p_8 = 647$	$\leftarrow$	$7 = q_8$,	then	$T(647)$	$=$	7
7)	$p_9 = 13$	$\rightarrow$	$701 = q_9$,	then	$T(13)$	$=$	701
8)	$p_{10} = 10939$	$\leftarrow$	$11 = q_{10}$,	then	$T(10939)$	$=$	11
9)	$p_{11} = 17$	$\rightarrow$	$4337 = q_{11}$,	then	$T(17)$	$=$	4337
10)	$p_{12} = 20029$	$\leftarrow$	$13 = q_{12}$,	then	$T(20029)$	$=$	13
11)	$p_{13} = 19$	$\rightarrow$	$32987 = q_{13}$,	then	$T(19)$	$=$	32987
12)	$p_{14} = 15649$	$\leftarrow$	$17 = q_{14}$,	then	$T(15649)$	$=$	17
13)	$p_{15} = 23$	$\rightarrow$	$276079 = q_{15}$,	then	$T(23)$	$=$	276079
14)	$p_{16} = 887459$	$\leftarrow$	$19 = q_{16}$,	then	$T(887459)$	$=$	19
15)	$p_{17} = 29$	$\rightarrow$	$207029 = q_{17}$,	then	$T(29)$	$=$	207029

etc.

Example 3.

1)	$p_3 = 3$	$\rightarrow$	$59 = q_3$,	then	$T(3)$	$=$	59
2)	$p_4 = 19$	$\leftarrow$	$3 = q_4$,	then	$T(19)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$37 = q_5$,	then	$T(5)$	$=$	37
4)	$p_6 = 13$	$\leftarrow$	$5 = q_6$,	then	$T(13)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$607 = q_5$,	then	$T(7)$	$=$	607
6)	$p_8 = 1831$	$\leftarrow$	$7 = q_8$,	then	$T(1831)$	$=$	7
7)	$p_9 = 11$	$\rightarrow$	$43 = q_9$,	then	$T(11)$	$=$	43
8)	$p_{10} = 179$	$\leftarrow$	$11 = q_{10}$,	then	$T(179)$	$=$	11
9)	$p_{11} = 17$	$\rightarrow$	$7489 = q_{11}$,	then	$T(17)$	$=$	7489
10)	$p_{12} = 39733$	$\leftarrow$	$13 = q_{12}$,	then	$T(39733)$	$=$	13
11)	$p_{13} = 23$	$\rightarrow$	$84991 = q_{13}$,	then	$T(23)$	$=$	84991
12)	$p_{14} = 56857$	$\leftarrow$	$17 = q_{14}$,	then	$T(56857)$	$=$	17
13)	$p_{15} = 29$	$\rightarrow$	$29789 = q_{15}$,	then	$T(29)$	$=$	29789
14)	$p_{16} = 88747$	$\leftarrow$	$19 = q_{16}$,	then	$T(88747)$	$=$	19
15)	$p_{17} = 31$	$\rightarrow$	$308927 = q_{17}$,	then	$T(31)$	$=$	308927

etc.

Example 4.

1)	$p_3 = 3$	$\rightarrow$	$67 = q_3$,	then	$T(3)$	$=$	67
2)	$p_4 = 11$	$\leftarrow$	$3 = q_4$,	then	$T(11)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$13 = q_5$,	then	$T(5)$	$=$	13
4)	$p_6 = 173$	$\leftarrow$	$5 = q_6$,	then	$T(173)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$47 = q_5$,	then	$T(7)$	$=$	47
6)	$p_8 = 1367$	$\leftarrow$	$7 = q_8$,	then	$T(1367)$	$=$	7
7)	$p_9 = 13$	$\rightarrow$	$461 = q_9$,	then	$T(13)$	$=$	461
8)	$p_{10} = 83$	$\leftarrow$	$11 = q_{10}$,	then	$T(83)$	$=$	11
9)	$p_{11} = 17$	$\rightarrow$	$5393 = q_{11}$,	then	$T(17)$	$=$	5393
10)	$p_{12} = 1801$	$\leftarrow$	$17 = q_{12}$,	then	$T(1801)$	$=$	17
11)	$p_{13} = 19$	$\rightarrow$	$51347 = q_{13}$,	then	$T(19)$	$=$	51347
12)	$p_{14} = 9403$	$\leftarrow$	$19 = q_{14}$,	then	$T(9403)$	$=$	19
13)	$p_{15} = 23$	$\rightarrow$	$1007359 = q_{15}$,	then	$T(23)$	$=$	1007359
14)	$p_{16} = 1419799$	$\leftarrow$	$23 = q_{16}$,	then	$T(1419799)$	$=$	23
15)	$p_{17} = 29$	$\rightarrow$	$2799941 = q_{17}$,	then	$T(29)$	$=$	2799941

etc.

Example 5.

1)	$p_3 = 3$	$\rightarrow$	$83 = q_3$,	then	$T(3)$	$=$	83
2)	$p_4 = 19$	$\leftarrow$	$3 = q_4$,	then	$T(19)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$13 = q_5$,	then	$T(5)$	$=$	13
4)	$p_6 = 53$	$\leftarrow$	$5 = q_6$,	then	$T(53)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$31 = q_5$,	then	$T(7)$	$=$	31
6)	$p_8 = 463$	$\leftarrow$	$7 = q_8$,	then	$T(463)$	$=$	7
7)	$p_9 = 11$	$\rightarrow$	$1291 = q_9$,	then	$T(11)$	$=$	1291
8)	$p_{10} = 523$	$\leftarrow$	$11 = q_{10}$,	then	$T(523)$	$=$	11
9)	$p_{11} = 13$	$\rightarrow$	$5501 = q_{11}$,	then	$T(13)$	$=$	5501
10)	$p_{12} = 3001$	$\leftarrow$	$17 = q_{12}$,	then	$T(3001)$	$=$	17
11)	$p_{13} = 17$	$\rightarrow$	$29401 = q_{13}$,	then	$T(17)$	$=$	29401
12)	$p_{14} = 20507$	$\leftarrow$	$19 = q_{14}$,	then	$T(20507)$	$=$	19
13)	$p_{15} = 23$	$\rightarrow$	$190543 = q_{15}$,	then	$T(23)$	$=$	190543
14)	$p_{16} = 342319$	$\leftarrow$	$23 = q_{16}$,	then	$T(342319)$	$=$	23
15)	$p_{17} = 29$	$\rightarrow$	$419189 = q_{17}$,	then	$T(29)$	$=$	419189

etc.

Example 6.

1)	$p_3 = 3$	$\rightarrow$	$107 = q_3$,	then	$T(3)$	$=$	107
2)	$p_4 = 19$	$\leftarrow$	$3 = q_4$,	then	$T(19)$	$=$	3
3)	$p_5 = 5$	$\rightarrow$	$109 = q_5$,	then	$T(5)$	$=$	109
4)	$p_6 = 29$	$\leftarrow$	$5 = q_6$,	then	$T(29)$	$=$	5
5)	$p_7 = 7$	$\rightarrow$	$79 = q_5$,	then	$T(7)$	$=$	79
6)	$p_8 = 311$	$\leftarrow$	$7 = q_8$,	then	$T(311)$	$=$	7
7)	$p_9 = 11$	$\rightarrow$	$523 = q_9$,	then	$T(11)$	$=$	523
8)	$p_{10} = 67$	$\leftarrow$	$11 = q_{10}$,	then	$T(67)$	$=$	11
9)	$p_{11} = 13$	$\rightarrow$	$22901 = q_{11}$,	then	$T(13)$	$=$	22901
10)	$p_{12} = 9277$	$\leftarrow$	$13 = q_{12}$,	then	$T(9277)$	$=$	13
11)	$p_{13} = 17$	$\rightarrow$	$91873 = q_{13}$,	then	$T(17)$	$=$	91873
12)	$p_{14} = 57737$	$\leftarrow$	$17 = q_{14}$,	then	$T(57737)$	$=$	17
13)	$p_{15} = 23$	$\rightarrow$	$345511 = q_{15}$,	then	$T(23)$	$=$	345511
14)	$p_{16} = 384547$	$\leftarrow$	$19 = q_{16}$,	then	$T(384547)$	$=$	19
15)	$p_{17} = 31$	$\rightarrow$	$563183 = q_{17}$,	then	$T(31)$	$=$	563183

etc.

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