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Equational theories of P-compatible varieties

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We consider a given type of algebras $\tau : F \to \mathbb{N}$, where F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of non-negative integers. Let $Id(\tau)$ be the set of all identities of type τ. For a term p of type τ different from a variable we denote by $ex(p)$ the most external fundamental operation symbol in p, i.e. if $f \in F$ and $p_1, \ldots, p_{\tau(f)}$ are terms of type τ then $ex(f(p_1, \ldots, p_{\tau(f)}) \equiv f$.

Let Π_F be the set of all partitions of F and let $P \in \Pi_F$. The block of P containing $f \in F$ will be denoted by $[f]_P$.

An identity $p = q$ of type τ is P-compatible if it is of the form $x = x$ or both p and q are not variables and $ex(p) \in [ex(q)]_P$.

So, if p and q are different from a variable, then $p = q$ is P-compatible if the outermost operation symbols in p and q belong to the same block of P.

The notion of P-compatible identity was introduced by J. Płonka in [9] and is a generalization of an externally compatible identity introduced by W. Chromik in [2] and a normal identity defined independently by J. Płonka [8] and I. I. Melnik [11].

An identity $p = q$ is externally compatible if it is of the form $x = x$ or $ex(p)$, $ex(q)$ are the same symbols.

So, $p = q$ is externally compatible if and only if it is P-compatible, where P consists of singletons only. We denote this partition of F by Ex and the partition $\{F\}$ by N. An identity $p = q$ is normal if it is of the form $x = x$ or neither p or q is a variable.

So, an identity is normal if and only if it is N-compatible.

We denote the set of all P-compatible identities of type τ by $P(\tau)$. So, $Ex(\tau)$ is the set of all externally compatible identities of type τ and $N(\tau)$ is the set of all normal identities of type τ.

If K is a variety then $P(K)$ denotes the set of all P-compatible identieties of type τ which are satisfied in K, $Id\,K$ the set of all identities satisfied in K.

We have: $P(K) = Id\,K \cap P(\tau)$, $N(K) = Id\,K \cap N(\tau)$ and $Ex(K) = Id\,K \cap Ex(\tau)$.

By K_P we denote the variety defined by $P(K)$.

For a set $\Sigma \subseteq Id(\tau)$ we denote by $D(\Sigma)$ the deductive closure of Σ (see Definition 14.1.6, p.94 in [1]), i.e. $D(\Sigma)$ is the smallest subset of $Id(\tau)$ containing Σ such that

(1) $x = x \in D(\Sigma)$ for every variable x;

(2) $p = q \in D(\Sigma) \Rightarrow q = p \in D(\Sigma)$;

(3) $p = q, q = r \in D(\Sigma) \Rightarrow p = r \in D(\Sigma)$;

(4) $D(\Sigma)$ is closed under replacement, that means, given any $p = q \in D(\Sigma)$ and any term r of type τ, if p occurs as a subterm of r, then by letting s be the result of replacing that occurence of p by q, we have $r = s \in D(\Sigma)$;

(5) $D(\Sigma)$ is closed under substitution, which means, for each $p = q \in D(\Sigma)$ and each term r of type, if we replace every ocurrence of a given variable x in $p = q$ by r, then the resulting identity belongs to $D(\Sigma)$.

By $Mod\,\Sigma$ we denote the set of all models of Σ, that is, the set of all algebras of type τ satisfying the identities from Σ. So, if $\Sigma = P(K)$ then $Mod\,\Sigma = K_P$.

A variety K such that $Id(K) = P(K)$ is called P-compatible. Let $\mathcal{L}(K)$ be the lattice of all subvarieties of K. The aim of this paper is to describe the lattice $\mathcal{L}(K_{Ex})$, where K is an idempotent variety.

It is well known that the lattice of equational theories of type τ is dually isomorphic to the lattice of all varieties of the same type and for equational theory Σ the lattice $\mathcal{L}(\Sigma)$ of all equational theories extending Σ is dually isomorphic to the lattice $\mathcal{L}(Mod\,\Sigma)$. So, we described also the lattice $\mathcal{L}(Ex(\tau))$ and $\mathcal{L}(Ex(K))$.

The concept of a P-compatible identity is related to the special structure of terms occurring in the identity. This structure is preserved by operator D and we can concider equational theories determined by P-compatible identities. It is easy to observe that $Id(\tau)$, $Ex(\tau)$, $N(\tau)$, $P(\tau)$ for $P \in \Pi_F$, $P(K)$ for a variety K are equational theories with respect to operator D. Now, we prove some lemmas which characterize this theories. We are use them to find the lattice $\mathcal{L}(Ex(\tau))$.

The lemmas and all theorems we give without proofs, because the full text of this paper will be published elsewhere.

Lemma 1. *Let $P_1, P_2 \in \Pi_F$. Then*

$$P_1 \subseteq P_2 \text{ if and only if } P_1(\tau) \subseteq P_2(\tau).$$

Lemma 2. *For any $P_1, P_2 \in \Pi_F$ the following conditions are equivalent*

(i) $P_1 = P_2$.

(ii) $P_1(\tau) = P_2(\tau)$.

This lemma follows immediately from lemma 1.

Lemma 3. *The identities (f, g):*

$$(f,g) \quad f(x_1, \ldots, x_{\tau(f)}) = g(x_{\tau(f)+1}, \ldots, x_{\tau(f)+\tau(g)}) \text{ for } f, g \in F$$

and $g \in [f]_P$ form an equational base for $P(\tau)$.

An immediate corollary from lemma 3 is the next lemma.

Lemma 4. *The following identities (f) for $f \in F$:*

$$(f)\; f(x_1, \ldots, x_{\tau(f)}) = f(x_{\tau(f)+1}, \ldots, x_{2\tau(f)})$$

form an equational base for $Ex(\tau)$.

This lemma was also noticed in W. Chromik [2].

Lemma 5. *Let Σ be an equational theory extending $Ex(\tau)$ and $\Sigma \neq Id(\tau)$. Then there exists exactly one partition P of F such that $P(\tau) = \Sigma$.*

Theorem 1. *The lattice $\mathcal{L}(Ex(\tau))$ is isomorphic to the lattice $\Pi_F + 1$ of all partitions of F with the additional greatest element 1.*

We called a variety K of type τ F-normal if and only if for any $f, g \in F$ there exists in $Id\,K$ an identity $p = q$ such that $ex(p) \equiv f$ and $ex(q) \equiv g$.

The aim of this section is to characterize the lattice $\mathcal{L}(Ex(K))$ of equational theories of F-normal variety K. Let us denote by T the trivial variety of type τ. Then $Id\,T = Id(\tau)$ and $P(T) = P(\tau)$ for $P \in \Pi_F$.

As an immediate consequence of Theorem 1 and the well-known fact that the lattice $\mathcal{L}(\Sigma)$ of all equational theories extending Σ is dually isomorphic to the lattice of subvarieties of the variety $Mod\,\Sigma$, we have the next lemma.

Lemma 6. $V \in \mathcal{L}(T_{Ex})$ *if and only if there exists a partition $P \in \Pi_F$ such that $V = T_P$ or $V = T$.*

Lemma 7. *Let K be a variety of type τ and $P \in \Pi_F$. Then $K_P = K \vee T_P$.*

Lemma 8. *If K is an F-normal variety of type τ and $P_1, P_2 \in \Pi_F$ then*

$$P_1 \neq P_2 \text{ if and only if } P_1(K) \neq P_2(K).$$

Theorem 2. *If K is an F-normal variety of type τ then the function*

$$\varphi : \Pi_F + 1 \to \mathcal{L}(Ex(K))$$

defined as follows $\varphi(1) = Id(\tau)$ and $\varphi(P) = P(K)$ for $P \in \Pi_F$ is a lattice embedding.

In W. Chromik and K. Hałkowska [3] the lattice $\mathcal{L}(D_{Ex})$ of subvarieties of the variety D_{Ex} defined by all externally compatible identities of the variety D of distributive lattices was described. It was proved that $\mathcal{L}(D_{Ex})$ is isomorphic to the direct product of a 2-element chain and a 3-element chain. In K. Hałkowska [7] the lattice $\mathcal{L}(B_{Ex})$ was described for the variety B of Boole'an algebras.

A variety K is called idempotent if and only if for any $f \in F$ the identity $f(x, \ldots, x) = x$ belongs to K.

Lemma 9. *Let K be an idempotent variety of type τ with $F \neq \emptyset$. Then for every variety $V \in \mathcal{L}(K_{Ex})$ the following identity holds:*

$$V = (V \cap K) \vee (V \cap T_{Ex}).$$

Lemma 10. *If K is an idempotent variety of type τ, $V_1 \in \mathcal{L}(K)$ and $V_2 \in \mathcal{L}(T_{Ex})$ then $(V_1 \vee V_2) \cap K = V_1$.*

Lemma 11. *If K is an idempotent variety of type τ $V_1 \in \mathcal{L}(K)$ and $V_2 \in \mathcal{L}(T_{Ex})$ then $(V_1 \vee V_2) \cap T_{Ex} = V_2$.*

The next lemma is some reformulation of the well known results of G. Grätzer [5].

Lemma 12. *Let K be a variety of type τ. The mapping*

$$\varphi : \mathcal{L}(K_{Ex}) \to \mathcal{L}(K) \times \mathcal{L}(T_{Ex})$$

defined for $V \in L(K_{Ex})$ as follows: $\varphi(V) = (V \cap K, V \cap T_{Ex})$ is a lattice isomorphism if and only if the following conditions are satisfied:

(1) $V = (V \cap K) \vee (V \cap T_{Ex})$,

(2) $(V_1 \vee V_2) \cap K = V_1$,

(3) $(V_1 \vee V_2) \cap T_{Ex} = V_2$

for $V \in \mathcal{L}(K_{Ex})$, $V_1 \in \mathcal{L}(K)$ and $V_2 \in \mathcal{L}(T_{Ex})$.

Theorem 3. *If K is an idempotent variety of type τ then the lattice $\mathcal{L}(K_{Ex})$ is isomorpic to the direct product of $\mathcal{L}(K)$ and $\mathcal{L}(T_{Ex})$.*

Proof. We define the mapping $\varphi : L(K_{Ex}) \to L(K) \times L(T_{Ex})$ as follows

$$\varphi(V) = (V \cap K, V \cap T_{Ex})$$

for $V \in L(K_{Ex})$. From lemmas 9, 10, 11 and 12 it follows that φ sets up a lattice isomorphism.

Corollary 1. *If K is a variety of idempotent grupoids then $\mathcal{L}(K_{Ex}) \cong \mathcal{L}(K) \times \mathbf{2}$, where $\mathbf{2}$ is a 2-element chain.*

Corollary 2. *If K is the variety of dstributive lattices then $\mathcal{L}(K_{Ex}) \cong \mathbf{2}\times\mathbf{3}$, where $\mathbf{2}$ is a 2-element chain and $\mathbf{3}$ is a 3-element chain.*

Corollary 2 is a theorem in W. Chromik and K. Hałkowska [3].

Let us note that the assumption in theorem 3 is essential. In K. Hałkowska [7] the variety $\mathcal{L}(B_{Ex})$ for the variety B of Boolean algebras was described. In this case the lattice $\mathcal{L}(B) \times \mathcal{L}(T_{Ex})$ is a proper sublattice of $\mathcal{L}(B_{Ex})$.

Theorem 3 can be applied also to a variety of algebras in which every operation is a projection.

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