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DISCRETE MATHEMATICS SCRIPT

Discrete mathematics

for computer science students

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UNIWERSYTET JANA DŁUGOSZA W CZĘSTOCHOWIE

Jan Długosz University in Czestochowa

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Notations

We will use the following notations:

$\mathbb{R}$ – set of real numbers,

$\mathbb{Q}$ – set of rational numbers,

$\mathbb{Z}$ – set of integers,

$\mathbb{N}$ – set of natural numbers (starting from 1),

$\mathbb{N}_0$ – set of natural numbers including 0,

$\mathbb{P}$ – set of prime numbers,

$|X|$ – number of elements in the set X,

$\mathcal{P}(X)$, 2^X – family of all subsets of the set X,

$\mathcal{D}(a)$ – set of divisors of the integer a, where $a \in \mathbb{Z}$,

$m \mid a$ – m divides a, where $m, a \in \mathbb{Z}$,

$m \nmid a$ – m does not divide a, where $m, a \in \mathbb{Z}$.

Greek alphabet:

A	α	alfa	N	ν	ni
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omikron
Δ	δ	delta	Π	π	pi
E	ϵ, ε	epsilon	P	ρ, ϱ	ro
Z	ζ	zeta	Σ	σ, ς	sigma
H	η	eta	T	τ	tau
Θ	θ, ϑ	theta	Υ	υ	ypsilon
I	ι	jota	Φ	ϕ, φ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mi	Ω	ω	omega

Preface

Discrete mathematics is a part of mathematics that deals with discrete structures. The term "discrete" should be understood here in the sense of "not continuous" or "separated from each other" (as opposed to "discreet" meaning "unobtrusive"). In a more restrictive sense, the term "discrete" is also understood as "finite", indicating that the objects of interest in discrete mathematics are finite structures and processes. Thus, discrete mathematics is fundamentally different from calculus, theory of differential equations, or topology, which are mainly concerned with continuous concepts and infinite objects.

More specifically, discrete mathematics is based on logic, set theory, and number theory, with its main branches including combinatorics and graph theory. It is applied in areas at the intersection of mathematics and computer science such as algorithms, cryptography, coding theory, and computational theory. Discrete mathematics is essential for understanding the theoretical foundations of computer science. A deep understanding of discrete mathematics allows for efficient solving of complex computing problems and the creation of effective algorithms.

This textbook aims to introduce the key concepts of discrete mathematics to all those interested in this field, particularly first-year undergraduate students in computer science. By learning the fundamentals of discrete mathematics, students develop problem-solving skills applicable in various real-life scenarios.

The material for the script "Discrete Mathematics for Computer Science Students" is selected to organize knowledge in this field acquired in high school and to supplement it with topics necessary for further studies in computer science. This script covers the basics of logic (Chapter 1), elements of number theory (Chapter 2), and an introduction to combinatorics (Chapter 3). Each chapter is supplemented with a set of exercises, most of which also include answers. The entire work is complemented by sample exam questions (Chapter 4).

Chapter 1

Elements of Logic

In this chapter, we will present selected elements of logic such as the basics of propositional calculus, predicate calculus, set theory, and relation theory. We will also introduce the notation for sums and products along with formulas useful for applying this notation. The following chapters briefly discuss the principle of mathematical induction and basic integer functions: the floor and ceiling functions. A large part of the content in this chapter is a repetition of knowledge from secondary school and will likely not pose problems for the reader. The exceptions may be the subsections concerning relations and mathematical induction, which are topics usually not covered in high school.

1.1. Basics of Propositional Calculus

In classical logic, we distinguish between two truth values: true and false. The symbol for true is 1, and the symbol for false is 0. We will start with a basic definition and examples that illustrate it.

Definition 1.1. A logical statement is any statement to which a truth value can be assigned: true or false.

Example 1.2. Examples of logical statements include:

1. $9 - 3 = 5$ (false).
2. $3 + 4 > 6$ (true).
3. Every square is a rectangle (true).
4. The number $\sqrt{3}$ is a rational number (false).
5. Nicea is the capital of France (false).
6. Mieszko I was the king of Spain (false).
7. The Earth has exactly one moon (true).
8. Kangaroos live in Antarctica (false).

Example 1.3. Examples of statements that are not logical statements:

1. Questions: Do crocodiles live in Africa?
2. Imperative sentences: Clean your room!

It should be noted that there are statements that are logical statements, even though at this moment, or given our current state of knowledge, we are unable to assess their truth value. For example, a statement like: "There are bacteria living on Mars." We cannot determine the truth value of this statement because our knowledge about Mars does not allow us to do so. However, it is a logical statement whose truth value we may be able to assess in the future.

Logical statements can be combined to form compound statements. To build compound statements, we use logical connectives (operators). The most commonly used logical connectives are summarized in Table 1.1. The truth value of a compound statement depends on the truth values of its component statements and the connectives used to form it. Negation is a unary operator that changes the truth value of the negated statement to its opposite (see Table 1.2). The remaining operators are binary, and the truth values of compound statements formed using them are presented in Table 1.3.

Table 1.1: Logical Connectives

symbol	name	usage	how to read
$\neg$, $\sim$	negation (negation)	$\neg p$, $\sim p$	not p
$\vee$	disjunction	$p \vee q$	p or q
$\veebar$	exclusive disjunction	$p \veebar q$	p either q
$\wedge$	conjunction	$p \wedge q$	p and q, p as well as q
$\Rightarrow$	implication	$p \Rightarrow q$	if p, then q
$\Leftrightarrow$	biconditional	$p \Leftrightarrow q$	p if and only if and only if q

In the implication "$p \Rightarrow q$", the statement p is called the **antecedent** of the implication, and the statement q is called the **consequent** of the implication.

Table 1.2: Negation

p	$\neg p$
1	0
0	1

Table 1.3: Binary Connectives

p	q	$p \vee q$	$p \veebar q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	0	0	0	1	1
0	1	1	1	0	1	0
1	0	1	1	0	0	0
1	1	1	0	1	1	1

Example 1.4. We will determine the truth values of the following compound logical statements:

1. It is not true that the Earth is flat.

   Let us denote the statement "The Earth is flat" by the symbol p. The statement p is false, thus based on Table 1.2, the statement $\neg p$ is true.

2. In Africa, there are black-and-white zebras and pink hippos.

   The above statement is written in natural language. To assess the truth value of this statement, we should change its form to a more precise one: "In Africa, there are black-and-white zebras and in Africa, there are pink hippos." Let us denote the statement "In Africa, there are black-and-white zebras" by the symbol p, and the statement "In Africa, there are pink hippos" by the symbol q. The statement p is true, and the statement q is false; thus (Table 1.3, column 5, row 4) the conjunction $p \wedge q$ is false.

3. If owls are mammals, then butterflies are insects.

   Let us denote the statement "Owls are mammals" by the symbol p, and the statement "Butterflies are insects" by the symbol q. The statement p is false, and the statement q is true, hence (Table 1.3, column 6, row 3) the implication $p \Rightarrow q$ is true.

4. $3 \mid 10 \Leftrightarrow (2+3) > 4$

   Let us denote the statement "$3 \mid 10$" by p, and the statement "$(2+3) > 4$" by the symbol q. The statement p is false, and the statement q is true; hence (Table 1.3, column 7, row 3) the statement $p \Leftrightarrow q$ is false.

The symbols p and q in tables 1.1, 1.2, and 1.3 (p. 7) are called **propositional variables**, which are variables under which we can substitute any logical statements. Expressions constructed from propositional variables and logical connectives (and, optionally, brackets) are called **formulas** or **statements of propositional calculus**.

Definition 1.5. A logical law (**law of propositional calculus**, **tautology**) is a formula of propositional calculus that is always true, meaning it is true when substituting any logical statements for the propositional variables.

We will present selected tautologies along with their names:

1. Law of double negation:
$$\neg(\neg p) \Leftrightarrow p,$$

2. Law of excluded middle:
$$p \vee \neg p,$$

3. Laws of associativity for disjunction and conjunction:
$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r), \quad (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r),$$

4. Laws of commutativity for disjunction and conjunction:
$$p \vee q \Leftrightarrow q \vee p, \quad p \wedge q \Leftrightarrow q \wedge p,$$

5. Laws of distribution of conjunction over disjunction and disjunction over conjunction:
$$(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r), \quad (p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r),$$

6. Law of transitivity of implication:
$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r),$$
7. Law of negation of implication:
$$\neg(p \Rightarrow q) \Leftrightarrow [p \wedge (\neg q)],$$
8. Law of transposition:
$$(p \Rightarrow q) \Leftrightarrow [(\neg q) \Rightarrow (\neg p)],$$
9. De Morgan's laws:
$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q), \qquad \neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q),$$
10. Rule of detachment:
$$[p \wedge (p \Rightarrow q)] \Rightarrow q,$$
11. Law of equivalence substitution:
$$(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q \wedge q \Rightarrow p).$$

Example 1.6. We will present one of the methods for proving that a given formula is a tautology. Consider the law of transitivity of implication:
$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r).$$
In the above formula, we have three logical variables: p, q, and r. We will list all possible values of these variables in a table in subsequent rows, as shown below:

p	q	r
0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
1	1	0
1	1	1

Next, in the headers of the subsequent columns, we will write all increasingly complex formulas that form the law of transitivity of implication:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
0	0	0					
0	0	1					
0	1	0					
1	0	0					
0	1	1					
1	0	1					
1	1	0					
1	1	1					

Finally, we fill in the table according to the rules of operation of logical connectives (see Tables 1.2 and 1.3, p. 7). Notice that in the last column, where we have the truth values of the law of transitivity of implication for all possible values of the variables p, q, and r, there are only truth symbols, which means that the given statement is a tautology.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	1	0	1
1	0	0	0	1	0	0	1
0	1	1	1	1	1	1	1
1	0	1	0	1	1	0	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

1.2. Basics of Quantifier Calculus

This subsection will begin with the introduction of the concept of a propositional function. A **propositional function** (also known as a **propositional formula**, **propositional form**, or **predicate**) of the variable x is an expression that contains the variable x and becomes a logical statement when we substitute an element from a certain set, which we call the domain or range of variability of the variable x, in place of x. In a similar way, we define propositional functions for a greater number of variables. For example, the expression "x is a positive number" is a propositional function, which becomes a true statement when we substitute any real number greater than zero for x, such as 5 or π, and becomes a false statement otherwise.

Quantifiers are another element of mathematical language that significantly expands the range of possible statements. They are phrases that allow the creation of new propositional functions and logical statements from already existing propositional functions. The two most commonly used quantifiers are:

- The universal quantifier, also called the large quantifier. This quantifier is used for phrases such as "for every," "for any," or "for all." The symbolic notation for this quantifier typically uses the symbols: $\bigwedge$ or $\forall$.
- The existential quantifier, also known as the small quantifier. This quantifier is used for phrases such as "there exists" or "for some." In symbolic notation, it usually takes one of the forms $\bigvee$ or $\exists$.

You may also encounter in the literature the notation $\exists!$, which expresses the quantifier: "there exists exactly one."

A quantifier always appears with a variable, which we say is a bound variable under the action of that quantifier. Variables that appear in a propositional function and are not bound by the action of a quantifier are called free variables. If a propositional function contains no quantifiers, then all variables present in it are free. If there is only one free variable in a propositional function, applying a binding quantifier to that function will yield a logical statement. Applying a quantifier to a propositional function with more free variables results in a new propositional function, but with fewer free variables.

Let $\Phi(x)$ be a propositional formula in which x is a free variable. Then:

1. The expression $\bigwedge\limits_{x} \Phi(x)$ can be read in one of the following ways:
   - For every x, $\Phi(x)$ holds,
   - For any x, $\Phi(x)$ holds,
   - For all x, $\Phi(x)$ holds,

2. The expression $\bigvee_{x} \Phi(x)$ can be read in one of the following ways:
   - There exists an x such that $\Phi(x)$ holds,
   - There exists an x for which $\Phi(x)$ holds,
   - For some x, $\Phi(x)$ holds.

To explicitly specify the range of variability of the variable, it is convenient to include the relevant information next to the quantifier. Thus:

- Instead of $\bigwedge_{x}(x \in X \wedge \Phi(x))$, we usually write $\bigwedge_{x\in X} \Phi(x)$,
- Instead of $\bigvee_{x}(x \in X \wedge \Phi(x))$, we usually write $\bigvee_{x\in X} \Phi(x)$.

Example 1.7. We will present several logical statements using quantifiers along with their corresponding symbolic notation:

1. For every natural number n, the number $n(n+1)$ is divisible by 2:
$$\bigwedge_{n\in\mathbb{N}} 2 \mid n(n+1).$$
2. There exists an integer x such that $x+7=-3$:
$$\bigvee_{x\in\mathbb{Z}} x+7=-3.$$
3. The square of every real number in the interval $(-1,1)$ is non-negative and less than 1:
$$\bigwedge_{x\in(-1,1)} 0 \leq x^2 < 1.$$
4. The sum of the squares of two arbitrary real numbers is a non-negative number:
$$\bigwedge_{x\in\mathbb{R}} \bigwedge_{y\in\mathbb{R}} x^2+y^2 \geq 0.$$
5. There exist natural numbers a, b, c, d such that $a^b = c^d$:
$$\bigvee_{a\in\mathbb{N}} \bigvee_{b\in\mathbb{N}} \bigvee_{c\in\mathbb{N}} \bigvee_{d\in\mathbb{N}} a^b = c^d.$$
6. For every integer x, there exists an integer y such that their sum is 0:
$$\bigwedge_{x\in\mathbb{Z}} \bigvee_{y\in Z} x+y=0.$$

Instead of several quantifiers of the same type standing next to each other, we can write one:

- Instead of: $\bigwedge_{x\in\mathbb{R}} \bigwedge_{y\in\mathbb{R}} x^2 + y^2 \geq 0$, we can write: $\bigwedge_{x,y\in\mathbb{R}} x^2 + y^2 \geq 0$,
- Instead of: $\bigvee_{a\in\mathbb{N}} \bigvee_{b\in\mathbb{N}} \bigvee_{c\in\mathbb{N}} \bigvee_{d\in\mathbb{N}} a^b = c^d$, we can write: $\bigvee_{a,b,c,d\in\mathbb{N}} a^b = c^d$.

We will present selected laws of quantifier calculus along with their names:

1. Law of rearrangement of universal quantifiers:

$$\bigwedge_{x\in X} \bigwedge_{y\in Y} \Phi(x, y) \iff \bigwedge_{y\in Y} \bigwedge_{x\in X} \Phi(x, y),$$

2. Law of rearrangement of existential quantifiers:

$$\bigvee_{x\in X} \bigvee_{y\in Y} \Phi(x, y) \iff \bigvee_{y\in Y} \bigvee_{x\in X} \Phi(x, y),$$

3. De Morgan's laws:

$$\neg \bigwedge_{x} \Phi(x) \iff \bigvee_{x} \neg\Phi(x), \qquad \neg \bigvee_{x} \Phi(x) \iff \bigwedge_{x} \neg\Phi(x),$$

4. Law of rearrangement of a universal quantifier with an existential quantifier:

$$\bigvee_{x\in X} \bigwedge_{y\in Y} \Phi(x, y) \Longrightarrow \bigwedge_{y\in Y} \bigvee_{x\in X} \Phi(x, y).$$

Remark 1.8. In point 4 above, the implication in the reverse direction does not hold. Consider the statement 6 from example 1.7 (p. 11):

$$\bigwedge_{x\in\mathbb{Z}} \bigvee_{y\in\mathbb{Z}} x + y = 0.$$

This statement is true because for every integer x, the number $y = -x$, chosen individually for each x, satisfies the condition $x + y = 0$. Rearranging the quantifiers gives us the statement:

$$\bigvee_{y\in\mathbb{Z}} \bigwedge_{x\in\mathbb{Z}} x + y = 0,$$

which is false because there is no integer y that, when added to any integer x, would always yield the same sum equal to 0.

1.3. Elements of Set Theory

A set is a fundamental concept in set theory and is uniquely defined by its elements, meaning there is exactly one set composed of given elements. When considering sets, it is necessary to indicate whether a given element belongs or does not belong to a specific set. We use the symbols $\in$ and $\notin$ for this purpose:

- $a \in A$ is read as "a is an element of the set A" or "a belongs to the set A",
- $b \notin A$ is read as "b is not an element of the set A" or "b does not belong to the set A".

Typically, sets are denoted by uppercase letters, while their elements are denoted by lowercase letters. There are several ways to define a set. One of them is to list all its elements. For example, the set denoted as A, consisting of odd numbers greater than 0 and less than 10, is presented as follows:

$$A = \{1, 3, 5, 7, 9\}.$$

In the case of sets with a large number of elements, this solution is impractical. Moreover, for sets with an infinite number of elements, this solution is impossible to apply. Therefore, a much better way to define a set is to construct a propositional function that is true only for its elements. In this way, the above set A can be defined as follows:

$$A = \{x \in \mathbb{N} : x = 2k + 1 \ \wedge \ k \in \mathbb{N}_0 \ \wedge \ x > 0 \ \wedge \ x < 10\}$$

or

$$A = \{x \in \mathbb{N} : x = 2k + 1 \ \wedge \ k \in \mathbb{N}_0 \ \wedge \ k \geq 0 \ \wedge \ k \leq 4\}.$$

If we wanted to define the set B consisting of odd numbers from a much larger range, for example from 0 to 2^{10}, listing all elements of the set B would be very time-consuming. One way is to use an ellipsis in the notation along with the initial and final elements of the set B:

$$B = \{1, 3, 5, \ldots, 1021, 1023\}.$$

However, this type of notation should be used with caution because the correct interpretation of the elements of the set largely depends on the assumptions of the reader. Meanwhile, following the definition of set A, we obtain:

$$B = \{x \in \mathbb{N} : x = 2k + 1 \ \wedge \ k \in \mathbb{N}_0 \ \wedge \ x > 0 \ \wedge \ x < 2^{10}\}$$

or

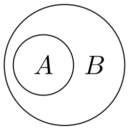
$$B = \{x \in \mathbb{N} : x = 2k + 1 \ \wedge \ k \in \mathbb{N}_0 \ \wedge \ k \geq 0 \ \wedge k \leq 2^9 - 1\}.$$

Notice that there exists a set that has no elements. This is the empty set, denoted by the symbol $\varnothing$, which can be defined as $\varnothing = \{\}$ or $\varnothing = \{x : x \neq x\}$.

When considering sets, it is often necessary to define a certain relation between two sets, as stated in the following definition.

Definition 1.9. We say that a set A is a **subset** of a set B if every element of set A is also an element of set B.

The above concept is denoted as $A \subseteq B$ and is read as "set A is a subset of set B" or "set A is contained in set B" or "set B contains set A." We can also illustrate it:



If $A \subseteq B$ and $B \subseteq A$, then obviously $A = B$. Furthermore, the concept of a subset can also be expressed symbolically as:

$$A \subseteq B \iff \bigwedge_{x \in A} (x \in A \Rightarrow x \in B).$$

Example 1.10. A specific example where $A \subseteq B$ is given by the sets A and B defined at the beginning of subsection 1.3 (p. 13).

When considering the concept of a subset, it is important to mention the power set.

Definition 1.11. The family of all subsets of a set X is called the **power set** of set X and is denoted by $\mathcal{P}(X)$ or 2^X.

The concept of a power set can be briefly summarized with the notation:

$$A \in \mathcal{P}(X) \iff A \subseteq X.$$

Example 1.12. For the set $X = \{a, b, c\}$, the power set $\mathcal{P}(X)$ is given by

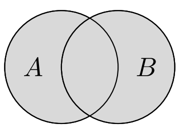
$$\mathcal{P}(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Notice that $|X| = 3$ and $|\mathcal{P}(X)| = 2^3 = 8$. We encourage the reader to experimentally investigate whether a four-element set will have $2^4 = 16$ elements in its power set. What is the general formula for the number of elements in a power set? It is worth comparing this example with Exercise 3.24 (p. 67).

The next step is to discuss operations that can be performed on sets.

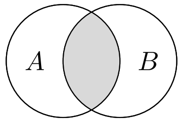
Definition 1.13. The **union** (or multiset union) of sets A and B, denoted $A \cup B$, is the set of elements that belong to set A or set B:

$$x \in A \cup B \iff x \in A \vee x \in B.$$



Definition 1.14. The **intersection** (or common part) of sets A and B, denoted $A \cap B$, is the set of elements that belong to both set A and set B:

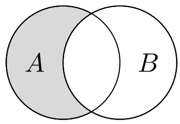
$$x \in A \cap B \iff x \in A \wedge x \in B,$$



Sets A and B are called **disjoint** if $A \cap B = \varnothing$.

Definition 1.15. The **difference** of sets A and B, denoted $A \setminus B$, is the set of elements that belong to set A and do not belong to set B:

$$x \in A \setminus B \iff x \in A \wedge x \notin B,$$



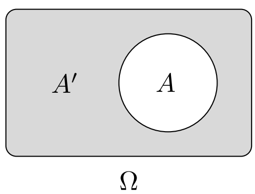
Example 1.16. If $A = \{a, b, c, d\}$ i $B = \{c, d, e, f\}$, then:

$$A \cup B = \{a, b, c, d, e, f\}, \quad A \setminus B = \{a, b\}, \quad A \cap B = \{c, d\}, \quad B \setminus A = \{e, f\}.$$

If all considered sets are subsets of a certain set Ω, which is called the space or universe, we can talk about the complements of sets.

Definition 1.17. If $A \subseteq \Omega$, the **complement** of set A in Ω is defined as the set $\Omega \setminus A$.

Typically, the complement of set A is denoted by A', and we can illustrate it as follows:



Example 1.18. Let $\Omega = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c, d\}$, and $B = \{c, d, e, f\}$. Then:

$$A' = \{e, f, g, h\}, \qquad B' = \{a, b, g, h\}.$$

We will present selected laws of set theory along with their names:

1. Law of double complementation:

$$(A')' = A,$$

2. Laws of associativity for union and intersection:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C),$$

3. Laws of commutativity for union and intersection:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A,$$

4. Laws of distribution of intersection over union and union over intersection:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \quad (A \cap B) \cup C = (A \cup C) \cap (B \cup C),$$

5. De Morgan's laws:

$$(A \cap B)' = A' \cup B', \quad (A \cup B)' = A' \cap B'.$$

Notice that De Morgan's laws have appeared three times: in classical propositional calculus, in quantifier calculus, and in set theory.

The last concept in this subsection is the Cartesian product.

Definition 1.19. The **Cartesian product** of non-empty sets A and B is defined as:

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

The Cartesian product of two sets is thus the set of all such pairs—two-element tuples—in which the first element of the pair belongs to the first set, while the second element of the pair is an element of the second set.

We can easily generalize the concept of the Cartesian product to any finite number of sets:

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i, i \in \{1, \ldots, n\}\}, \ n \in \mathbb{N}.$$

Example 1.20.

1. For the sets $A = \{1, 4, 5\}$ and $B = \{2, 3\}$, we have

$$A \times B = \{(1, 2), (1, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}.$$

2. If $A = \{3, 7\}$ and $B = \{7, 9\}$, then:

$$A \times B = \{(3, 7), (3, 9), (7, 7), (7, 9)\}.$$

3. For $A = \{2, 6\}$, $B = \{w, z\}$, and $C = \{\alpha, \beta\}$, we have

$$\begin{aligned} &A \times B \times C = \\ &\quad = \{(2, w, \alpha), (2, w, \beta), (2, z, \alpha), (2, z, \beta), (6, w, \alpha), (6, w, \beta), (6, z, \alpha), (6, z, \beta)\}. \end{aligned}$$

We encourage the reader to consider the relationship between the number of elements in the Cartesian product and the cardinalities of the sets forming that product. We will return to this issue in subsection 3.3, p. 59.

Remark 1.21.

1. When defining sets, we always use curly braces. Furthermore, each element of a set "occurs" in it exactly once, which means that the set $\{a, a, b, c, c, c\}$ is identical to the set $\{a, b, c\}$. The order in which we list the elements of a set also does not matter, so the same set can also be written as $\{b, c, a\}$.

2. The elements of the Cartesian product—tuples—are written using parentheses. It is important to pay attention to the order of the elements in the tuple because if $a \neq b$, then $(a, b) \neq (b, a)$. Consequently, the Cartesian product is not commutative: if $A \neq B$, then $A \times B \neq B \times A$.

1.4. Elements of Relation Theory

The study of relationships between objects, phenomena, and concepts is the essence of science. The same is true in mathematics. However, we often use the term "relation" without considering the deeper meaning of the word. For instance, in this textbook, in the paragraph just before definition 1.9 (p. 14), we used the phrase "relation between two sets" (this case will be discussed in detail in example 1.31, p. 19). Beyond mathematics, various relations are considered, although the concept of a relation usually remains informal. In this regard, mathematics differs from other fields, as it must rely on precise concepts and formal definitions.

Definition 1.22. Let X be a non-empty set. Any subset $R \subseteq X \times X$ is called a **binary relation** in the set X.

In other words, a relation in the set X is a subset of the Cartesian product $X \times X$. If $(x, y) \in R$, we say that the element x **is in relation** R with the element y, which we will denote as xRy or $R(x, y)$.

Example 1.23. Let $X = \{1, 2, 5, 8\}$. In the set X, we can define many relations in various ways.

1. We can list all the elements of the relation:

$$R_1 = \{(1, 5), (8, 5), (2, 5), (8, 2), (8, 8)\}.$$

2. We can define the relation by providing a propositional function that is true only for its elements:

$$R_2 = \{(a, b) \in X \times X : a + b \text{ is an odd number}\},$$

which is convenient when the set X has many elements. In our example, X has only 4 elements, so we can afford to list all the elements of the relation R_2:

$$R_2 = \{(1, 2), (2, 1), (1, 8), (8, 1), (2, 5), (5, 2), (5, 8), (8, 5)\}.$$

3. We can specify which elements of the set X are in relation by providing a propositional function on the right side of the equivalence that is true for the elements in relation:

$$R_3(a, b) \iff a \mid b \quad \text{for } a, b \in X.$$

The relation defined above consists of the following pairs:

$$R_3 = \{(1, 1), (1, 2), (1, 5), (1, 8), (2, 2), (2, 8), (5, 5), (8, 8)\}.$$

Among the many interesting properties that relations can have, the most important are: reflexivity, symmetry, antisymmetry, and transitivity. These will allow us to define two extremely important types of relations: partial orders and equivalence relations.

Definition 1.24. Let X be a non-empty set. We say that a relation R in the set X is:

1. **reflexive** if $\bigwedge_{x \in X} xRx$,
2. **symmetric** if $\bigwedge_{x,y \in X} xRy \Rightarrow yRx$,
3. **transitive** if $\bigwedge_{x,y,z \in X} xRy \wedge yRz \Rightarrow xRz$,
4. **antisymmetric** if $\bigwedge_{x,y \in X} xRy \wedge yRx \Rightarrow x = y$.

Definition 1.25. Let X be a non-empty set. We say that a relation R in the set X is a **equivalence relation** if it is reflexive, symmetric, and transitive.

The concept of an equivalence class is intrinsically linked to an equivalence relation, which is the set of all elements with which a given element is related.

Definition 1.26. Let R be an equivalence relation in a non-empty set X and let $x \in X$. The set

$$[x]_R = \{y \in X : xRy\}$$

is called the **equivalence class** of the element x.

Remark 1.27. The set $[x]_R$ is always non-empty because, due to the reflexivity of the relation R, we have $x \in [x]_R$. It is also easy to observe that if two elements are related, then their equivalence classes are equal. Conversely, if two elements are not related, then their equivalence classes are disjoint.

Example 1.28. Let us examine the following relation in the set $\mathbb{Z}$:

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 5 \mid (a - b)\}.$$

The relation R is:

1. reflexive, because for any $a \in \mathbb{Z}$, we have:
$$5 \mid 0 \;\Rightarrow\; 5 \mid (a - a) \;\Rightarrow\; aRa.$$
2. symmetric, because for any $a, b \in \mathbb{Z}$, we have:
$$aRb \;\Rightarrow\; 5 \mid (a - b) \;\Rightarrow\; 5 \mid [-(a - b)] \;\Rightarrow\; 5 \mid (b - a) \;\Rightarrow\; bRa.$$
3. transitive, because for any $a, b, c \in \mathbb{Z}$, if aRb and bRc, that is, $5 \mid (a - b)$ and $5 \mid (b - c)$, then $a - b = 5k$ and $b - c = 5l$ for some $k, l \in \mathbb{Z}$. Thus,
$$a - c = (a - b) + (b - c) = 5k + 5l = 5(k + l),$$
and therefore:
$$5 \mid (a - c) \;\Rightarrow\; aRc.$$

These considerations justify that the relation R is an equivalence relation.

Let's check what the equivalence class $[0]_R$ looks like. We are looking for integers b such that $5 \mid (0-b)$. Notice that:

$$5 \mid (0-b) \;\Rightarrow\; 5 \mid (-b) \;\Rightarrow\; 5 \mid b.$$

Thus, the equivalence class $[0]_R$ includes all integers divisible by 5.

Now, let's examine which elements belong to the equivalence class $[1]_R$. We are looking for integers b for which $5 \mid (1-b)$, which means $1-b = 5k$ for some $k \in \mathbb{Z}$. Hence, $b = 1-5k = 1+5(-k)$, so the equivalence class $[1]_R$ includes all integers that leave a remainder of 1 when divided by 5.

Similar considerations can be made for $[2]_R$, $[3]_R$, and $[4]_R$. Notice that $5 \in [0]_R$, $6 \in [1]_R$, $7 \in [2]_R$, etc., leading to the conclusion that every integer belongs to one of the 5 equivalence classes determined by the remainders when divided by 5:

$$\begin{aligned}
[0]_R &= \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\},\\
[1]_R &= \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\},\\
[2]_R &= \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\},\\
[3]_R &= \{\ldots, -7, -2, 3, 8, 13, 18, \ldots\},\\
[4]_R &= \{\ldots, -6, -1, 4, 9, 14, 19, \ldots\}.
\end{aligned}$$

Definition 1.29. Let X be a non-empty set. We say that a relation R in the set X is a **partial order relation** (is a partial order) if it is reflexive, transitive, and antisymmetric.

Example 1.30. The relation R from example 1.28 (p. 18) is not a partial order relation because it is not antisymmetric. It is possible to find such $a, b \in \mathbb{Z}$ that $5 \mid (a-b)$ and $5 \mid (b-a)$, but $a \neq b$. For example, let $a = 12$ and $b = 27$.

Example 1.31. Let Z be a set such that $|Z| \geq 2$. Let's examine the relation of set inclusion, that is, the relation R in the power set $\mathcal{P}(Z)$ defined as:

$$R(K, L) \iff K \subseteq L \quad \text{for } K, L \in \mathcal{P}(Z).$$

This relation is:

1. reflexive, because for any $K \in \mathcal{P}(Z)$ we have
$$K \subseteq K \;\Rightarrow\; R(K, K).$$

2. transitive, because for any $K, L, M \in \mathcal{P}(Z)$ we have
$$R(K, L) \wedge R(L, M) \;\Rightarrow\; K \subseteq L \wedge L \subseteq M \;\Rightarrow\; K \subseteq M \;\Rightarrow\; R(K, M).$$

3. antisymmetric, because for any $K, L \in \mathcal{P}(Z)$ we have:
$$R(K, L) \wedge R(L, K) \;\Rightarrow\; K \subseteq L \wedge L \subseteq K \;\Rightarrow K = L.$$

These considerations justify that R is a partial order relation. The inclusion relation is not an equivalence relation because it is not symmetric: for example, taking $K = \varnothing$ and letting L be any non-empty subset of Z, we have $K \subseteq L$, but $L \not\subseteq K$.

In a partial order relation, we may encounter a situation where many elements are incomparable, meaning that both x is not related to y and y is not related to x. If we require that such a situation does not occur, meaning that any two elements are comparable, we obtain a linear order.

Definition 1.32. We say that a relation R in a non-empty set X is a **linear order relation** if it is a partial order relation and:

$$\bigwedge_{x,y\in X} xRy \vee yRx.$$

Example 1.33. The relation from example 1.31 (p. 19) is not a linear order relation. The set Z has at least two elements, so let us take such z_1, $z_2 \in Z$ that $z_1 \neq z_2$. Of course, $\{z_1\}$ and $\{z_2\} \in \mathcal{P}(Z)$, but $\{z_1\} \not\subseteq \{z_2\} \wedge \{z_2\} \not\subseteq \{z_1\}$.

Example 1.34. Let $X = \mathbb{R}$. Consider the following relation:

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}.$$

The reader can easily verify that this is a partial order relation. It is also a linear order relation because for any real numbers $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$.

1.5. Notation for Sums and Products

To concisely represent the sum of many terms, we use the summation symbol, which is the uppercase letter Σ (sigma). Below and above this symbol (or in the lower and upper indices), we indicate the range of summation:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

An alternative notation for the above sum can take the form:

$$\sum_{k=1}^{n} a_k = \sum_{k\in\{1,2,\ldots,n\}} a_k = \sum_{1\leq k\leq n} a_k.$$

Similarly, to concisely represent the product of many factors, we use the multiplication symbol Π (pi):

$$\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdots a_{n-1} \cdot a_n.$$

The above product can also be written as:

$$\prod_{k=1}^{n} a_k = \prod_{k\in\{1,2,\ldots,n\}} a_k = \prod_{1\leq k\leq n} a_k.$$

The fundamental properties useful when working with the summation symbol are as follows:

1. $\displaystyle\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k, \;\; \lambda \in \mathbb{R},$

2. $\displaystyle\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + b_k), \quad \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k - b_k),$

3. $\displaystyle\sum_{k=1}^{n} a_k = \sum_{k=1+t}^{n+t} a_{k-t},$

4. $\displaystyle\sum_{k=1}^{n} a_k + \sum_{k=n+1}^{m} a_k = \sum_{k=1}^{m} a_k,$

5. $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{r} a_{k,j} = \sum_{j=1}^{r}\sum_{k=1}^{n} a_{k,j},$

6. $\displaystyle\sum_{k=1}^{n} a_k \cdot \sum_{j=1}^{r} b_j = \sum_{k=1}^{n}\sum_{j=1}^{r}(a_k b_j) = \sum_{j=1}^{r}\sum_{k=1}^{n}(a_k b_j).$

The fundamental properties of the multiplication symbol are as follows:

1. $\displaystyle\prod_{k=1}^{n} a_k^{\lambda} = \left(\prod_{k=1}^{n} a_k\right)^{\lambda}, \;\; \lambda \in \mathbb{R},$

2. $\displaystyle\prod_{k=1}^{n} a_k \cdot \prod_{k=1}^{n} b_k = \prod_{k=1}^{n}(a_k \cdot b_k), \quad \prod_{k=1}^{n} a_k \div \prod_{k=1}^{n} b_k = \prod_{k=1}^{n}(a_k \div b_k),$

3. $\displaystyle\prod_{k=1}^{n} a_k = \prod_{k=1+t}^{n+t} a_{k-t},$

4. $\displaystyle\prod_{k=1}^{n} a_k \cdot \prod_{k=n+1}^{m} a_k = \prod_{k=1}^{m} a_k,$

5. $\displaystyle\prod_{k=1}^{n}\prod_{j=1}^{r} a_{k,j} = \prod_{j=1}^{r}\prod_{k=1}^{n} a_{k,j}.$

It can be easily observed that the summation properties from 1 to 5 have analogous versions for multiplication. However, the summation property number 6 does not have a corresponding counterpart for multiplication. Let's take a closer look at this and write down the property for $n = 2$ and $r = 2$:

$$\sum_{k=1}^{2} a_k \cdot \sum_{j=1}^{2} b_j = (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2 = \sum_{k=1}^{2}\sum_{j=1}^{2}(a_k b_j).$$

If an analogous property held for multiplication, its left and right sides would look as follows and would be equal:

$$L = \prod_{k=1}^{2} a_k + \prod_{j=1}^{2} b_j = (a_1 \cdot a_2) + (b_1 \cdot b_2),$$
$$P = \prod_{k=1}^{2} \prod_{j=1}^{2} (a_k + b_j) = (a_1 + b_1) \cdot (a_1 + b_2) \cdot (a_2 + b_1) \cdot (a_2 + b_2).$$

But, the equality $L = P$ is false because in the set of real numbers, addition does not distribute over multiplication. To verify this, it is sufficient to take $a_1 = a_2 = b_1 = b_2 = 1$.

Example 1.35. Let us expand the following expressions:

1. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5,$

2. $\prod_{k=1}^{4} (b_k + 2) = (b_1 + 2) \cdot (b_2 + 2) \cdot (b_3 + 2) \cdot (b_4 + 2),$

3. $\sum_{k=3}^{6} 7c_k = 7c_3 + 7c_4 + 7c_5 + 7c_6,$

4. $\prod_{k=1}^{4} d_{2k}^3 = d_2^3 \cdot d_4^3 \cdot d_6^3 \cdot d_8^3,$

5. $\sum_{k=1}^{3} \sum_{j=2}^{4} e_{k,j} = \sum_{k=1}^{3} (e_{k,2} + e_{k,3} + e_{k,4}) = e_{1,2} + e_{1,3} + e_{1,4} + e_{2,2} + e_{2,3} + e_{2,4} + {} + e_{3,2} + e_{3,3} + e_{3,4},$

6. $\sum_{k=2}^{5} (-1)^k f_k = f_2 - f_3 + f_4 - f_5.$

1.6. Mathematical Induction

Mathematical induction is a method for proving statements about natural numbers. In its basic version, it relies on the application of the following theorem.

Theorem 1.36. *Let $T(n)$ be a logical statement for $n \in \mathbb{N}$. If:*

1. *the statement $T(1)$ is true,*
2. *for every $k \in \mathbb{N}$, if the statement $T(k)$ is true, then the statement $T(k+1)$ is also true,*

then the statement $T(n)$ is true for every $n \in \mathbb{N}$.

The first condition is called the **base case**. The second condition, known as the **inductive step**, involves using the **inductive hypothesis** to prove the **inductive conclusion**.

Example 1.37. Using mathematical induction, we will prove that $6 \mid (n^3 - n)$ for $n \in \mathbb{N}$.

1. Base case: For $n = 1$, we have $n^3 - n = 1^3 - 1 = 0$, and the statement $6 \mid 0$ is true.
2. Inductive step. Let us fix an arbitrary $k \in \mathbb{N}$.
   - Inductive hypothesis: $6 \mid (k^3 - k)$.
   - Inductive conclusion: $6 \mid ((k+1)^3 - (k+1))$.

   We have

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 = \\ &= (k^3 - k) + (3k^2 + 3k) = \\ &= (k^3 - k) + 3k(k+1).\end{aligned}$$

   The expression $k^3 - k$ is divisible by 6 by the inductive hypothesis. The factors of the product $3k(k+1)$ are two consecutive natural numbers: k and $k+1$, so one of them is even, and therefore the product is also even. Moreover, this product is a multiple of 3, so it is divisible by 6. Hence, $(k+1)^3 - (k+1)$, as a sum of numbers divisible by 6, is also divisible by 6, which completes the proof of the inductive conclusion.

By virtue of theorem 1.36 (p. 22): $6 \mid (n^3 - n)$ for every $n \in \mathbb{N}$.

Example 1.38. We will prove that $6 \mid (10^n + 4^n - 2)$ for $n \in \mathbb{N}$, using mathematical induction.

1. Base case: For $n = 1$, we have $10^n + 4^n - 2 = 10^1 + 4^1 - 2 = 12$ and $6 \mid 12$, thus the statement $6 \mid (10^1 + 4^1 - 2)$ is true.
2. Inductive step. Let us fix an arbitrary $k \in \mathbb{N}$.
   - Inductive hypothesis: $6 \mid (10^k + 4^k - 2)$.
   - Inductive conclusion: $6 \mid (10^{k+1} + 4^{k+1} - 2)$.

   We have

$$\begin{aligned}10^{k+1} + 4^{k+1} - 2 &= 10 \cdot 10^k + 4 \cdot 4^k - 2 = \\ &= 10 \cdot (10^k + 4^k - 2) + (18 - 6 \cdot 4^k) = \\ &= 10 \cdot (10^k + 4^k - 2) + 6(3 - 4^k).\end{aligned}$$

   The first term of the above sum is divisible by 6 by the inductive hypothesis, and the second term is a multiple of 6, so the entire sum is also divisible by 6.

By virtue of theorem 1.36 (p. 22): $6 \mid (10^n + 4^n - 2)$ for every $n \in \mathbb{N}$.

1.7. Integer-Valued Functions

In discrete mathematics, we like to work with integers, so functions that round real numbers to integers are useful:

1. $\lfloor x \rfloor$ – the largest integer less than or equal to x, called the **floor**, rounds real numbers down,

2. $\lceil x \rceil$ – the smallest integer greater than or equal to x, called the **ceiling**, rounds real numbers up.

The **fractional part** of a number is also related to the floor and is defined by the formula:

$$\langle x \rangle = x - \lfloor x \rfloor.$$

The fundamental properties of the floor and ceiling functions are as follows:

1. $\lfloor x \rfloor = c \Longleftrightarrow c \leqslant x < c+1$,
2. $\lfloor x \rfloor = c \Longleftrightarrow x-1 < c \leqslant x$,
3. $\lceil x \rceil = c \Longleftrightarrow c-1 < x \leqslant c$,
4. $\lceil x \rceil = c \Longleftrightarrow x \leqslant c < x+1$.

There are also many additional properties that are worth knowing:

1. $x-1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x+1$,
2. $0 \leq \langle x \rangle < 1$,
3. $\lfloor x \rfloor = \lceil x \rceil = x \Longleftrightarrow x \in \mathbb{Z}$,
4. $\lceil x \rceil - \lfloor x \rfloor = 1 \Longleftrightarrow x \notin \mathbb{Z}$,
5. $\lfloor -x \rfloor = -\lceil x \rceil$,
6. $\lceil -x \rceil = -\lfloor x \rfloor$,
7. $\lfloor x+k \rfloor = \lfloor x \rfloor + k \Longleftrightarrow k \in \mathbb{Z}$,
8. $\lceil x+k \rceil = \lceil x \rceil + k \Longleftrightarrow k \in \mathbb{Z}$,
9. $x < k \Longleftrightarrow \lfloor x \rfloor < k, \ k \in \mathbb{Z}$,
10. $k < x \Longleftrightarrow k < \lceil x \rceil, \ k \in \mathbb{Z}$,
11. $x \leqslant k \Longleftrightarrow \lceil x \rceil \leqslant k, \ k \in \mathbb{Z}$,
12. $k \leqslant x \Longleftrightarrow k \leqslant \lfloor x \rfloor, \ k \in \mathbb{Z}$.

Example 1.39. Let us compute the value of the expression:

$$\big\lfloor \lceil 5{,}1 \rceil - \langle 7{,}7 \rangle + \lceil -8{,}3 \rceil - \langle -9{,}6 \rangle \cdot \lfloor -2{,}5 \rfloor \big\rfloor = \big\lfloor 6 - 0{,}7 - 8 - 0{,}4 \cdot (-3) \big\rfloor = \lfloor -1{,}5 \rfloor = -2.$$

Example 1.40. Let us calculate how many integers are in the interval $[-55, 45]$ that are divisible by 6. We divide the interval $[-55, 45]$ into parts: $[-55, -1]$, $(-1, 1)$, $[1, 45]$. We introduce a working notation $d(A)$ for the number of integers divisible by 6 in the set A. Notice that $d([1, x]) = d([-x, -1]) = \lfloor x/6 \rfloor$ for any $x \in \mathbb{R}$. Thus, we have:

$$\begin{aligned} d([-55, 45]) &= d([-55, -1]) + d((-1, 1)) + d([1, 45]) \\ &= d([1, 55]) + 1 + d([1, 45]) = \\ &= \left\lfloor \frac{55}{6} \right\rfloor + 1 + \left\lfloor \frac{45}{6} \right\rfloor \\ &= 9 + 1 + 7 = 17. \end{aligned}$$

Let us verify the result by listing all these numbers:

$$-54, -48, -42, -36, -30, -24, -18, -12, -6, 0, 6, 12, 18, 24, 30, 36, 42.$$

1.8. Exercises

Exercise 1.1. Indicate which of the following statements are logical statements:

a) $2 + 2 = 10$.

b) Don't eat so many hamburgers!

c) There exists a value x such that $x^2 - 2x + 9 = 0$.

d) Who is the president of France?

e) $|5| = -5$.

f) The number π is rational.

g) Be careful!

h) How many continents are there on Earth?

i) All sharks are herbivorous.

j) Kraków was the capital of Poland.

k) Every quadrilateral is convex.

l) What is the highest peak in the Alps?

m) $3 > 7$.

n) Viruses live at the bottom of the Pacific Ocean.

Exercise 1.2. Determine the truth value of the following formulas, assuming the following truth values for the propositional variables: $p = 1, q = 0, r = 1$:

a) $[(p \vee q) \wedge r] \Leftrightarrow (p \wedge \neg q)$,

b) $[(p \Leftrightarrow q) \vee (p \Rightarrow r)] \Rightarrow (\neg \wedge p)$,

c) $[(p \Rightarrow q) \wedge \neg r] \Leftrightarrow \neg p$,

d) $[(p \vee q) \Rightarrow (p \vee \neg q)] \Leftrightarrow (\neg p \vee q)$,

e) $[(p \vee q) \wedge \neg p] \Rightarrow q$,

f) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (r \vee q)$.

Exercise 1.3. Determine the truth values of the following statements:

a) It is not true that $3 + 4 = 9$.

b) If $2 + 2 = 4$, then $2 + 4 = 8$.

c) If $2 + 2 = 5$, then $2 + 4 = 6$.

d) If $2 + 2 = 4$, then $2 + 4 = 6$.

e) If $2 + 2 = 5$, then $2 + 4 = 8$.

f) If the Earth has the shape of a cone, then Bolesław Chrobry was the first king of Poland.

g) If Washington was the first president of the United States, then $3 + 3 = 8$.

h) In Poland, coffee or cocoa is grown.

i) $(1 + 2)^2 \neq (-1 - 2)^2 \;\vee\; -0.5 > -\frac{1}{3}$.

j) $(15 \mid 45 \;\vee\; 5 \mid 45) \iff [(2 < -1) \implies (4^2 = (-4)^2)]$.

k) $[(2 + 3)^2 = 25 \iff (2 + 3)^2 > 1] \;\wedge\; [(2 \cdot 2 = 4) \iff (3 \cdot 5 = 10)]$.

l) The Sun is a star if and only if the Earth has three moons.

Exercise 1.4. Check whether the given formulas are tautologies:

a) $[(p \Leftrightarrow q) \vee (\neg p \wedge q)] \Rightarrow (p \vee q)$,

b) $[(p \vee \neg q) \Rightarrow \neg q] \wedge (p \Leftrightarrow q)$,

c) $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$,

d) $[(p \vee q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow q) \vee (q \Rightarrow r)]$,

e) $(p \wedge q) \vee [(\neg p \wedge q) \vee (\neg p \wedge \neg q)]$,

f) $(((p \wedge q) \Rightarrow r) \wedge (p \vee q) \Rightarrow \neg r) \Rightarrow (p \wedge q \wedge r)$.

Exercise 1.5. Create the negations of the following statements, using the appropriate laws (p. 8):

a) Marek spent his vacation in Greece or Spain.

b) Beata is studying French and English.

c) If Adam is a student, then he does not work.

d) 7 is a natural number or a prime number.

e) 2 is an even number or 5 is a divisor of 8.

f) 3 is not a composite number and 9 is not an even number.

g) Penguins do not fly and the elephant is larger than the goat.

h) If snow is white, then grass is pink.

Exercise 1.6. Determine the truth values of the following statements:

a) $\bigwedge_{x\in\mathbb{R}} |x| + 1 > 0$,

b) $\bigwedge_{x\in\mathbb{R}} \sin^2 x + \cos^2 x = 1$,

c) $\bigvee_{x\in\mathbb{R}} x^2 - x + 1 = 0$,

d) $\bigwedge_{a,b\in\mathbb{R}} \bigwedge_{n\in\mathbb{N}} (ab)^n = a^n b^n$,

e) $\bigwedge_{x\in\mathbb{R}} \bigvee_{y\in\mathbb{R}} x + y > 0$,

f) $\bigvee_{y\in\mathbb{R}} \bigwedge_{x\in\mathbb{R}} x + y > 0$,

g) $\bigvee_{x\in\mathbb{Z}} x > 5 \vee x < 3$,

h) $\bigvee_{x\in\mathbb{Z}} x > 5 \wedge x < 3$,

i) $\bigvee_{x\in\mathbb{N}} x > 7 \wedge x < 8$.

Exercise 1.7. Evaluate the truth value of the given statements and write them using quantifiers and mathematical symbols:

a) Every natural number is non-negative.

b) There exists a real number x such that $x + 5 = 12$.

c) The square of any real number increased by 1 is a positive number.

d) There exists an integer whose cube is a negative number.

e) For every real number, there exists an integer that is less than it.

f) There exists a real number that is not greater than any natural number.

g) There exists an integer that is not greater than any real number.

h) For every real number x, there exists a real number y such that $x - y$ is negative.

Exercise 1.8. Create the negations of the following statements, using the appropriate laws (p. 12):

a) All students passed the exam in discrete mathematics.

b) There is a person who knows their future.

c) Every real number is positive or every real number is negative.

d) There exists a natural number that is odd and divisible by 10.

e) All rectangles are squares.

f) There exists a natural number divisible by 3 and every integer is non-negative.

Exercise 1.9. List the elements of the following sets:

a) $A = \{x \in \mathbb{Z} : -4 < x \leq 3\}$,

b) $B = \{x \in \mathbb{N} : (x + 3)(x - 2) = 0\}$,

c) $C = \{x \in \mathbb{R} : |x| = 3\}$,

d) $D = \{x \in \mathbb{N} : x = 2k \wedge k \in \mathbb{Z} \wedge x < 8\}$,

e) $E = \{x \in \mathbb{N} : x \mid 12\}$,

f) $F = \{x \in \mathbb{Z} : 4 \mid x \ \wedge \ -7 < x \leq 4\}$.

Exercise 1.10. Provide all elements of the power set $\mathcal{P}(X)$ when:

a) $X = \{k, l\}$,

b) $X = \{\{a, b\}, 3\}$,

c) $X = \{\text{cat, horse, perch}\}$,

d) $X = \{a, \beta, C, \delta\}$,

e) $X = \{\{x, 4\}, \{y, 7\}\}$,

f) $X = \{\{K, \{L, M\}\}, \alpha\}$.

Exercise 1.11. Determine the sets $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$ if:

a) $A = \{a, b, c\}$, $B = \{c, d\}$,

b) $A = \{-2, -1, 0, 1, 5\}$, $B = \{1, 3, 5, 8\}$,

c) $A = \{5, 7, 9, 11\}$, $B = \{7, 9\}$,

d) $A = \{-1, -2, -3\}$, $B = \{1, 2, 3\}$,

e) $A = \{x \in N : x < 9\}$, $B = x \in N : x \geq 3$.

Exercise 1.12. Let the set $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be space. Determine the sets A', B', $A' \cup B'$, $(A \cup B)'$, $A' \cap B'$, and $(A \cap B)'$ if:

a) $A = \{x \in \Omega : x \text{ is a prime number}\}$, $B = \{x \in \Omega : x = 2k \ \wedge \ k \in \mathbb{N}\}$,

b) $A = \{x \in \Omega : x \mid 8\}$, $B = \{x \in \Omega : 2 \mid x\}$,

c) $A = \{x \in \Omega : 3 \mid x\}$, $B = \{x \in \Omega : x = 2k + 1 \ \wedge \ k \in \mathbb{N}_0\}$.

Exercise 1.13. Given the sets:

$$A = \{1, 2, 3\},\ B = \{3, 4\},\ C = \{5\},\ D = \{w, z\},\ E = \{w, x, y, z\}.$$

Determine the Cartesian products:

a) $A \times B$,
b) $B \times A$,
c) $B \times B$,
d) $A \times B \times C$,
e) $B \times C \times B$,
f) $D \times B$,
g) $E \times D$,
h) $C \times E$.

Exercise 1.14. Verify whether the given relation is reflexive, symmetric, antisymmetric, or transitive:

a) $X = \mathbb{N},\ R_\alpha = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \mid b\}$,

b) $X = \mathbb{Z},\ R_\beta = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \mid b\}$,

c) $X = \mathbb{N},\ R_1 = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x < y\}$,

d) $X = \mathbb{N},\ R_2 = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$,

e) $X = \mathbb{R},\ R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}$,

f) $X = \mathbb{N},\ R_4 = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y^2\}$,

g) $X = \mathbb{R},\ R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y^2\}$,

h) $X = \mathbb{Z},\ \bar{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3|(a - b)\}$,

i) $X = \mathbb{Z},\ \tilde{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 7|(a - b)\}$,

j) $X = \mathbb{R} \times \mathbb{R},\ \hat{R} = \{((x_1, y_1), (x_2, y_2)) \in (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) : x_1 \leq x_2 \wedge y_1 \leq y_2\}$,

k) $X = \mathcal{P}(\{a, b, c\}),\ T = \{(K, L) \in \mathcal{P}(\{a, b, c\}) \times \mathcal{P}(\{a, b, c\}) : K \subseteq L\}$,

l) X – set of points in the plane Oxy,
$H = \{(a, b) \in X \times X :$ the distance of point a from the origin is equal to the distance of point b from the origin$\}$,

m) X – set of all countries,
$G = \{(x, y) \in X \times X :$ country x has a land border with country $y\}$,

n) X – set of words in the English language dictionary,
$S_1 = \{(x, y) \in X \times X :$ word x has at least one common letter with word $y\}$,

o) X – set of words in the English language dictionary,
$S_2 = \{(x, y) \in X \times X :$ word x has the same number of letters as word $y\}$,

p) X – set of words in the English language dictionary,
$S_3 = \{(x, y) \in X \times X :$ word x starts with the same letter as word $y\}$,

q) X – set of words in the English language dictionary,
$S_4 = \{(x, y) \in X \times X :$ word x occupies an earlier or the same position in lexicographic order as word $y\}$,

r) X – set of lines in the plane,
$K_1 = \{(x, y) \in X \times X :$ line x is parallel to line $y\}$,

s) X – set of lines in the plane,
$K_2 = \{(a, b) \in X \times X : \text{line } a \text{ is perpendicular to line } b\}$,

t) X – set of Polish citizens living on January 1st 2024 at 00:01,
$P_1 = \{(k, l) \in X \times X : \text{person } k \text{ is an ancestor of person } l\}$,

u) X – set of Polish citizens living on January 1st 2024 at 00:01,
$P_2 = \{(k, l) \in X \times X : \text{person } k \text{ is a parent of person } l\}$,

v) X – set of Polish citizens living on January 1st 2024 at 00:01,
$P_3 = \{(k, l) \in X \times X : \text{person } k \text{ is a child of person } l\}$,

w) X – set of Polish citizens living on January 1st 2024 at 00:01,
$P_4 = \{(k, l) \in X \times X : \text{person } k \text{ is a husband or wife of person } l\}$,

x) X – set of Polish citizens living on January 1st, 2024, at 00:01,
$P_5 = \{(k, l) \in X \times X : \text{person } k \text{ has at least one common parent with person } l\}$,

y) X – set of Polish citizens living on January 1st, 2024, at 00:01,
$P_6 = \{(k, l) \in X \times X : \text{person } k \text{ has the same parents as person } l\}$,

z) X – set of Polish citizens living on January 1st, 2024, at 00:01,
$L = \{(k, l) \in X \times X : \text{person } k \text{ was born in the same month as person } l\}$.

Exercise 1.15. Indicate which relations from task 1.14 are:

a) equivalence relations; determine their equivalence classes,

b) partial orders,

c) linear orders.

Exercise 1.16. List all the elements of the following sums and products:

a) $\displaystyle\sum_{i=1}^{5} a_i^3$,

b) $\displaystyle\prod_{j=3}^{7} 4b_j$,

c) $\displaystyle\sum_{k=2}^{6} c_{k+1}$,

d) $\displaystyle\sum_{i=1}^{4} (-1)^i d_i e_i$,

e) $\displaystyle\sum_{j=2}^{5}\sum_{k=3}^{6} f_{j,k}$,

f) $\displaystyle\prod_{0\le m\le 4} g_{m^2}$,

g) $\displaystyle\sum_{1\le i,j\le 4} a_i b_j$,

h) $\displaystyle\sum_{1\le i<j\le 4} a_i b_j$,

i) $\displaystyle\sum_{1\le i\le j\le 4} a_i b_j$,

j) $\displaystyle\prod_{k=2}^{4} (c_k + d_k)$,

k) $\displaystyle\prod_{\substack{i,j\ge 1\\ i+j=5}} e_{i,j}$,

l) $\displaystyle\sum_{5\le m<8} (f_m + 4)$.

Exercise 1.17. Express the following expressions using summation or product notation:

a) $1 + 2 + 3 + 4 + \cdots + 17$,

b) $1 + 2 + 4 + 8 + \cdots + 512$,

c) $18 + 21 + 24 + 27 + \cdots + 45$,

d) $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots + \dfrac{1}{6561}$,

e) $2 \cdot 4 \cdot 6 \cdot 8 \cdots 26$,

f) $\dfrac{1}{5} \cdot \dfrac{1}{15} \cdot \dfrac{1}{25} \cdots \dfrac{1}{85}$,

g) $-1 + 2 - 3 + 4 - 5 + \cdots - 17$,

h) $1 - 2 + 3 - 4 + 5 - \cdots + 17$.

Exercise 1.18. Show that for every natural number n:

a) $3 \mid (n^3 + 2n)$,
b) $3 \mid (n^3 - 3n^2 + 2n - 3)$,
c) $6 \mid (n^3 - n)$,
d) $6 \mid (n^3 + 12n)$,
e) $6 \mid (n^3 + 3n^2 + 2n)$,
f) $9 \mid (n^3+(n+1)^3+(n+2)^3)$.

Exercise 1.19. Show that for every natural number n:

a) $4 \mid (5^{5n+3} + 3)$,
b) $7 \mid (2^{n+2} + 3^{2n+1})$,
c) $8 \mid (5^{2n+1} + 3)$,
d) $10 \mid (3^{4n+2} + 1)$,
e) $11 \mid (2^{6n+1} + 3^{2n+2})$,
f) $14 \mid (3^{4n+2} + 5^{2n+1})$.

Exercise 1.20. Calculate:

a) $\lceil 2\pi \rceil + \lceil \sqrt{150} \rceil$,

b) $\dfrac{\lceil \lceil -8.6 \rceil \cdot \lfloor 2.44 \rfloor + \langle -5.54 \rangle \rceil}{\lfloor \sqrt{300} \rfloor}$,

c) $\lceil \lceil 3.75 \rceil \cdot \langle 4.53 \rangle \rceil + \lceil 11.6 \rceil - 3 \cdot \lfloor -5.9 \rfloor$,

d) $\dfrac{\lceil \lceil 4.75 \rceil + \lfloor -3.41 \rfloor - \langle 2.43 \rangle \rceil}{\lfloor \lfloor \pi \rceil - \lceil 2^4 - 3.5 \rceil + \langle -7.33 \rangle \rangle}$.

Exercise 1.21. Using any real number x that is not an integer, verify the basic and complementary properties of integer-valued functions (p. 24).

Exercise 1.22.

a) How many numbers in the interval $[1, 30]$ are divisible by 7?

b) How many numbers in the interval $[0, 42]$ are divisible by 9?

c) How many numbers in the interval $[-33, -1]$ are divisible by 5?

d) How many numbers in the interval $[-25, 11]$ are divisible by 6?

e) How many numbers in the interval $[17, 53]$ are divisible by 8?

f) How many numbers in the interval $[1, 7564]$ are divisible by 7?

g) How many numbers in the interval $[0, 5437]$ are divisible by 11?

h) How many numbers in the interval $[-3563, -1]$ are divisible by 9?

i) How many numbers in the interval $[-2565, 3451]$ are divisible by 6?

j) How many numbers in the interval $[1787, 5663]$ are divisible by 7?

1.9. Answers

Answer 1.1. Logical statements: a, c, e, f, i, j, k, m, n.

Answer 1.2.

a) 1,
b) 1,
c) 1,
d) 0,
e) 1,
f) 1.

Answer 1.3.

a) 1, b) 0, c) 1, d) 1, e) 1, f) 1, g) 0, h) 0, i) 0, j) 1, k) 0, l) 0.

Answer 1.4.

a) No, the value is 0 for $(p, q) = (0, 0)$.

b) No, the value is 0 for $(p, q) \in \{(1, 0), (0, 1), (1, 1)\}$.

c) Yes.

d) No, the value is 0 for $(p, q, r) \in \{(0, 1, 0), (1, 0, 0), (1, 1, 0)\}$.

e) No, the value is 0 for $(p, q) = (1, 0)$.

f) No, the value is 0 for $(p, q, r) \in \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

Answer 1.5.

a) Marek did not spend his vacation in Greece and did not spend his vacation in Spain.

b) Beata is not studying French or is not studying English.

c) Adam is a student and works.

d) 7 is not a natural number and is not a prime number.

e) 2 is not an even number and 5 is not a divisor of 8.

f) 3 is a composite number or 9 is an even number.

g) Penguins fly or the elephant is not bigger than the goat.

h) Snow is white and the grass is not pink.

Answer 1.6.

a) 1, b) 1, c) 0, d) 1, e) 1, f) 0, g) 1, h) 0, i) 0.

Answer 1.7.

a) 1, $\displaystyle\bigwedge_{x\in\mathbb{N}} x \geq 0.$

b) 1, $\displaystyle\bigvee_{x\in\mathbb{R}} x + 5 = 12.$

c) 1, $\displaystyle\bigwedge_{x\in\mathbb{R}} x^2 + 1 > 0.$

d) 1, $\displaystyle\bigvee_{x\in\mathbb{Z}} x^3 < 0.$

e) 1, $\displaystyle\bigwedge_{x\in\mathbb{R}} \bigvee_{y\in\mathbb{Z}} y < x.$

f) 1, $\displaystyle\bigvee_{x\in\mathbb{R}} \bigwedge_{y\in\mathbb{N}} x \leq y.$

g) 0, $\displaystyle\bigvee_{x\in\mathbb{Z}} \bigwedge_{y\in\mathbb{R}} x \leq y.$

h) 1, $\displaystyle\bigwedge_{x\in\mathbb{R}} \bigvee_{y\in\mathbb{R}} x - y < 0.$

Answer 1.8.

a) There exists a student who did not pass the exam in discrete mathematics.

b) No person knows their future.

c) There exists a real number that is not positive, and there exists a real number that is not negative.

d) Every natural number is even or not divisible by 10.

e) There exists a rectangle that is not a square.

f) Every natural number is not divisible by 3 or there exists a negative integer.

Answer 1.9.

a) $A = \{-3, -2, -1, 0, 1, 2, 3\}$,

b) $B = \{2\}$,

c) $C = \{-3, 3\}$,

d) $D = \{2, 4, 6\}$,

e) $E = \{1, 2, 3, 4, 6, 12\}$,

f) $F = \{-4, 0, 4\}$.

Answer 1.10.

a) $\mathcal{P}(X) = \{\varnothing,\ \{k\},\ \{l\},\ \{k, l\}\}$,

b) $\mathcal{P}(X) = \{\varnothing,\ \{\{a, b\}\},\ \{3\},\ \{\{a, b\}, 3\}\}$,

c) $\mathcal{P}(X) = \{\varnothing,\ \{\text{cat}\},\ \{\text{horse}\},\ \{\text{perch}\},\ \{\text{cat, horse}\},\ \{\text{cat, perch}\},$ $\{\text{horse, perch}\},\ \{\text{cat, horse, perch}\}\}$,

d) $\mathcal{P}(X) = \{\varnothing,\ \{a\},\ \{\beta\},\ \{C\},\ \{\delta\},\ \{a, \beta\},\ \{a, C\},\ \{a, \delta\},\ \{\beta, C\},\ \{\beta, \delta\},\ \{C, \delta\},$ $\{a, \beta, C\},\ \{a, \beta, \delta\},\ \{a, C, \delta\},\ \{\beta, C, \delta\},\ \{a, \beta, C, \delta\}\}$,

e) $\mathcal{P}(X) = \{\varnothing,\ \{\{x, 4\}\},\ \{\{y, 7\}\},\ \{\{x, 4\}, \{y, 7\}\}\}$,

f) $\mathcal{P}(X) = \{\varnothing,\ \{\{K, \{L, M\}\}\},\ \{\alpha\},\ \{\{K, \{L, M\}\}, \alpha\}\}$.

Answer 1.11.

a) $A \cup B = \{a, b, c, d\},\ \ A \cap B = \{c\},\ \ A \setminus B = \{a, b\},\ \ B \setminus A = \{d\}$.

b) $A \cup B = \{-2, -1, 0, 1, 5, 3, 8\},\ \ A \cap B = \{1, 5\},\ \ A \setminus B = \{-2, -1, 0, \},\ \ B \setminus A = \{3, 8\}$.

c) $A \cup B = \{5, 7, 9, 11\},\ \ A \cap B = \{7, 9\},\ \ A \setminus B = \{5, 11\},\ \ B \setminus A = \varnothing$.

d) $A \cup B = \{-1, -2, -3, 1, 2, 3\},\ \ A \cap B = \varnothing,\ \ A \setminus B = A,\ \ B \setminus A = B$.

e) $A \cup B = \mathbb{N},\ \ A \cap B = \{3, 4, 5, 6, 7, 8\},\ \ A \setminus B = \{1, 2\},\ \ B \setminus A = \{x \in \mathbb{N} : x \geq 9\}$.

Answer 1.12.

a) $A = \{2, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$,
$A' = \{1, 4, 6, 8, 9\}$, $B' = \{1, 3, 5, 7, 9\}$,
$A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9\}$, $(A \cup B)' = \{1, 9\}$,
$A' \cap B' = \{1, 9\}$, $(A \cap B)' = \{1, 3, 4, 5, 6, 7, 8, 9\}$.

b) $A = \{1, 2, 4, 8\}$, $B = \{2, 4, 6, 8\}$,
$A' = \{3, 5, 6, 7, 9\}$, $B' = \{1, 3, 5, 7, 9\}$,
$A' \cup B' = \{1, 3, 5, 6, 7, 9\}$, $(A \cup B)' = \{3, 5, 7, 9\}$,
$A' \cap B' = \{3, 5, 7, 9\}$, $(A \cap B)' = \{1, 3, 5, 6, 7, 9\}$.

c) $A = \{3, 6, 9\}$, $B = \{1, 3, 5, 7, 9\}$,
$A' = \{1, 2, 4, 5, 7, 8\}$, $B' = \{2, 4, 6, 8\}$,
$A' \cup B' = \{1, 2, 4, 5, 6, 7, 8\}$, $(A \cup B)' = \{2, 4, 8\}$,
$A' \cap B' = \{2, 4, 8\}$, $(A \cap B)' = \{1, 2, 4, 5, 6, 7, 8\}$.

Answer 1.13.

a) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$,

b) $B \times A = \{(3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$,

c) $B \times B = \{(3, 3), (3, 4), (4, 3), (4, 4)\}$,

d) $A \times B \times C = \{(1, 3, 5), (1, 4, 5), (2, 3, 5), (2, 4, 5), (3, 3, 5), (3, 4, 5)\}$,

e) $B \times C \times B = \{(3, 5, 3), (3, 5, 4), (4, 5, 3), (4, 5, 4)\}$,

f) $D \times B = \{(w, 3), (w, 4), (z, 3), (z, 4)\}$,

g) $E \times D = \{(w, w), (w, z), (x, w), (x, z), (y, w), (y, z), (z, w), (z, z)\}$,

h) $C \times E = \{(5, w), (5, x), (5, y), (5, z)\}$.

Answer 1.14.

	refl.?	sym.?	trans.?	antisym.?	equiva.?	part. o.?	lin. o.?
a	Y	N	Y	Y		✓	
b	Y	N	Y	N			
c	N	N	Y	Y*			
d	Y	N	Y	Y		✓	✓
e	Y	N	Y	Y		✓	✓
f	Y	N	N	N			
g	N	N	N	N			
h	Y	Y	Y	N	✓		
i	Y	Y	Y	N	✓		
j	Y	N	Y	Y		✓	
k	Y	N	Y	Y		✓	
l	Y	Y	Y	N	✓		
m	Y	Y	N	N			
n	Y	Y	N	N			
o	Y	Y	Y	N	✓		
p	Y	Y	Y	N	✓		
q	Y	N	Y	Y		✓	✓
r	Y	Y	Y	N	✓		
s	N	Y	N	N			
Y	N	N	Y	Y*			
u	N	N	N	Y*			
v	N	N	N	Y*			
w	N	Y	N	N			
x	N	Y	N	N			
y	Y	Y	Y	N	✓		
z	Y	Y	Y	N	✓		

* – The antecedent of the implication is vacuously satisfied, the entire implication is true.

Answer 1.15.

a) Equivalence relations: h, i, l, o, p, r, y, z.

Equivalence classes for h:

- $[0]_{\bar{R}} = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$,
- $[1]_{\bar{R}} = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$,
- $[2]_{\bar{R}} = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$.

Equivalence classes for i:

- $[0]_{\tilde{R}} = \{\ldots, -14, -7, 0, 7, 14, \ldots\}$,
- $[1]_{\tilde{R}} = \{\ldots, -13, -6, 1, 8, 15, \ldots\}$,
- $[2]_{\tilde{R}} = \{\ldots, -12, -5, 2, 9, 16, \ldots\}$,
- $[3]_{\tilde{R}} = \{\ldots, -11, -4, 3, 10, 17, \ldots\}$,
- $[4]_{\tilde{R}} = \{\ldots, -10, -3, 4, 11, 18, \ldots\}$,
- $[5]_{\tilde{R}} = \{\ldots, -9, -2, 5, 12, 19, \ldots\}$,
- $[6]_{\tilde{R}} = \{\ldots, -8, -1, 6, 13, 20, \ldots\}$.

Equivalence classes for l: one equivalence class is the set of points on a circle centered at the point $(0, 0)$.

Equivalence classes for o: one equivalence class is the set of words that have the same number of letters.

Equivalence classes for p: one equivalence class is the set of words that start with the same letter.

Equivalence classes for r: one equivalence class is the set of lines that have the same direction.

Equivalence classes for y: one equivalence class is the set of individuals who have the same parents.

Equivalence classes for z: one equivalence class is the set of individuals born in the same month.

b) Partial order relations: a, d, e, j, k, q.

c) Linear order relations: d, e, q.

Answer 1.16.

a) $a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3$,

b) $4b_3 \cdot 4b_4 \cdot 4b_5 \cdot 4b_6 \cdot 4b_7$,

c) $c_3 + c_4 + c_5 + c_6 + c_7$,

d) $-d_1e_1 + d_2e_2 - d_3e_3 + d_4e_4$,

e) $f_{2,3}+f_{2,4}+f_{2,5}+f_{2,6}+f_{3,3}+f_{3,4}+f_{3,5}+f_{3,6}+f_{4,3}+f_{4,4}+f_{4,5}+f_{4,6}+f_{5,3}+f_{5,4}+f_{5,5}+f_{5,6}$,

f) $g_0 \cdot g_1 \cdot g_4 \cdot g_9 \cdot g_{16}$,

g) $a_1b_1 + a_1b_2 + a_1b_3 + a_1b_4 + a_2b_1 + a_2b_2 + a_2b_3 + a_2b_4 + a_3b_1 + a_3b_2 + a_3b_3 + a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4$,

h) $a_1b_2 + a_1b_3 + a_1b_4 + a_2b_3 + a_2b_4 + a_3b_4$,

i) $a_1b_1 + a_1b_2 + a_1b_3 + a_1b_4 + a_2b_2 + a_2b_3 + a_2b_4 + a_3b_3 + a_3b_4 + a_4b_4$,

j) $(c_2 + d_2) \cdot (c_3 + d_3) \cdot (c_4 + d_4)$,

k) $e_{1,4} \cdot e_{2,3} \cdot e_{3,2} \cdot e_{4,1}$,

l) $(f_5 + 4) + (f_6 + 4) + (f_7 + 4)$.

Answer 1.17.

a) $\displaystyle\sum_{k=1}^{17} k$, c) $\displaystyle\sum_{j=6}^{15} 3j$, e) $\displaystyle\prod_{i=1}^{13} 2i$, g) $\displaystyle\sum_{k=1}^{17} (-1)^k k$,

b) $\displaystyle\sum_{i=0}^{9} 2^i$, d) $\displaystyle\sum_{k=0}^{8} \frac{1}{3^k}$, f) $\displaystyle\prod_{j=0}^{8} \frac{1}{5(2j+1)}$, h) $\displaystyle\sum_{k=1}^{17} (-1)^{k+1} k$.

Answer 1.18. —

Answer 1.19. —

Answer 1.20.

a) 20, b) $-\dfrac{15}{17}$, c) 33, d) $-\dfrac{1}{10}$.

Answer 1.21. —

Answer 1.22.

a) $\left\lfloor \dfrac{30}{7} \right\rfloor = 4,$

b) $\left\lfloor \dfrac{42}{9} \right\rfloor + 1 = 5,$

c) $\left\lfloor \dfrac{33}{5} \right\rfloor = 6,$

d) $\left\lfloor \dfrac{25}{6} \right\rfloor + \left\lfloor \dfrac{11}{6} \right\rfloor + 1 = 6,$

e) $\left\lfloor \dfrac{53}{8} \right\rfloor - \left\lfloor \dfrac{16}{8} \right\rfloor = 4,$

f) $\left\lfloor \dfrac{7564}{7} \right\rfloor = 1080,$

g) $\left\lfloor \dfrac{5437}{11} \right\rfloor + 1 = 495,$

h) $\left\lfloor \dfrac{3563}{9} \right\rfloor = 395,$

i) $\left\lfloor \dfrac{2565}{6} \right\rfloor + \left\lfloor \dfrac{3451}{6} \right\rfloor + 1 = 1003,$

j) $\left\lfloor \dfrac{5663}{7} \right\rfloor - \left\lfloor \dfrac{1786}{7} \right\rfloor = 554.$

Chapter 2

Elements of Number Theory

In this chapter, we will present selected elements of number theory, such as divisibility of numbers, prime and composite numbers, greatest common divisor and least common multiple, the Euclidean algorithm, and relatively prime numbers. These concepts are already introduced in mathematics and computer science classes in primary and secondary school; however, it is very important to organize and supplement this knowledge.

2.1. Divisibility of Numbers

We will begin by recalling the basic definition related to divisibility. We say that an integer m **divides** an integer a, or equivalently, is a **divisor** of the integer a, if there exists an integer n such that $mn = a$. We denote this fact as $m \mid a$. If the number m does not divide the number a, we write $m \nmid a$. The set of all integer divisors of the number a is denoted by $\mathcal{D}(a)$. If the number a is natural, we distinguish **proper divisors** within its set of divisors. These are the divisors that are natural numbers different from a. Conventionally, we do not refer to proper divisors of negative numbers.

Example 2.1. Let $a = 12$ and $m = 3$. Then $3 \mid 12$, and the set of all integer divisors of 12 is as follows:

$$\mathcal{D}(12) = \{-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12\}.$$

Of course, $\mathcal{D}(12) = \mathcal{D}(-12)$. The proper divisors of the number 12 are 1, 2, 3, 4, and 6.

Let us recall that the divisibility relation in the set of natural numbers:

$$R_{\mathbb{N}} = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \mid b\}$$

is a partial order relation (Exercise 1.14a, p. 28). However, the analogous relation considered in the set of integers:

$$R_{\mathbb{Z}} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \mid b\}$$

is not antisymmetric (Exercise 1.14b, p. 28), therefore it is no longer a partial order relation.

The most important properties of divisibility are listed below. Let $a, b, m \in \mathbb{Z}$. Then:

1. if $m \mid a$ and $m \mid b$, then $m \mid (a+b)$ and $m \mid (a-b)$,
2. if $m \mid a$ and $a \mid b$, then $m \mid b$,
3. if $m \mid a$, then $m \mid ab$,
4. if $m \mid a$ and $a \neq 0$, then $|m| \leq |a|$,
5. if $m \mid a$ and $a \mid m$, then $m = a$ or $m = -a$,
6. $m \mid 0$, in particular, $0 \mid 0$,
7. for $a \neq 0$, it is not true that $0 \mid a$,
8. $1 \mid a, \quad -1 \mid a, \quad a \mid a, \quad -a \mid a$.

In the next step, we will present a theorem that forms the basis of the Euclidean algorithm.

Theorem 2.2. *If $a, b \in \mathbb{Z}$ and $b \neq 0$, then there exists exactly one pair of integers q, r satisfying the conditions:*

$$a = qb + r, \quad 0 \leq r < |b|. \tag{2.1}$$

Moreover, $b \mid a$ holds if and only if $r = 0$.

The number r from the statement of the above theorem is called the **remainder** of the division of a by b, and the number q is called the **quotient** of this division.

Example 2.3.

1. If $a = 31$, $b = 9$, then $q = 3$, $r = 4$, because $31 = 3 \cdot 9 + 4$.
2. If $a = 31$, $b = -9$, then $q = -3$, $r = 4$, because $31 = -3 \cdot (-9) + 4$.
3. If $a = -31$, $b = 9$, then $q = -4$, $r = 5$, because $-31 = (-4) \cdot 9 + 5$.
4. If $a = -31$, $b = -9$, then $q = 4$, $r = 5$, because $-31 = 4 \cdot (-9) + 5$.

Remark 2.4. It is essential to remember that the remainder r from the division is always a non-negative number. The quotient q is chosen in such a way that the equality (2.1) holds for the remainder r.

At the end of this subsection, let us recall the divisibility rules that allow for a quick check, without performing the division, whether a given number is divisible by another number. We will limit ourselves here to presenting the divisibility rules for a few numbers:

- $2 \mid n$ if the last digit of the number n is even.
- $3 \mid n$ if the sum of the digits of the number n is divisible by 3.
- $4 \mid n$ if the last two digits of the number n form a number that is divisible by 4.
- $5 \mid n$ if the last digit of the number n is 0 or 5.

- $7 \mid n$ if the alternating sum of the numbers formed by grouping the digits of the number n in threes (starting from the right) is divisible by 7.
- $8 \mid n$ if the last three digits of the number n form a number that is divisible by 8.
- $9 \mid n$ if the sum of the digits of the number n is divisible by 9.
- $11 \mid n$ if the alternating sum of the digits of the number n is divisible by 11.
- $13 \mid n$ if the alternating sum of the numbers formed by grouping the digits of the number n in threes (starting from the right) is divisible by 13.

Above, we intentionally omitted the numbers 6, 10, and 12. The divisibility rules for these numbers (as well as for 14 and 15) will be discussed at the end of section 2.3 on page 42.

Example 2.5.

1. Let $a = 6\,379\,586$. Then $2 \mid a$, because the last digit of a is 6, which is even.
2. Let $b = 2\,686\,863$. Then $3 \mid b$, because $2 + 6 + 8 + 6 + 8 + 6 + 3 = 39$ and $3 \mid 39$.
3. Let $c = 22\,738\,848$. Then $4 \mid c$, because the last two digits of c form the number 48, which is divisible by 4.
4. Let $d = 34\,567\,525$. Then $5 \mid d$, because the last digit of d is 5.
5. Let $f = 2\,172\,912$. Then $7 \mid f$, because $2 - 172 + 912 = 742$ and $7 \mid 742$.
6. Let $g = 77\,269\,936$. Then $8 \mid g$, because the last three digits of g form the number 936, which is divisible by 8.
7. Let $h = 7\,878\,258$. Then $9 \mid h$, because $7 + 8 + 7 + 8 + 2 + 5 + 8 = 45$ and $9 \mid 45$.
8. Let $j = 940\,412$. Then $11 \mid j$, because $9 - 4 + 0 - 4 + 1 - 2 = 0$ and $11 \mid 0$.
9. Let $l = 7\,207\,356$. Then $13 \mid l$, because $7 - 207 + 356 = 156$ and $13 \mid 156$.

2.2. Prime and Composite Numbers

Prime numbers have a very simple definition, which is already known to primary school students. Although they were known in antiquity, they continue to fascinate many professional mathematicians as well as enthusiasts not professionally involved in mathematics. For example, the extremely simple formulation of Goldbach's conjecture (actually proposed by Euler in 1742) remains unresolved—it has not been proven, nor has a counterexample been found to date.

Conjecture 2.6. (Goldbach's Conjecture) *Every even number greater than 2 is the sum of two (not necessarily distinct) prime numbers.*

Let us begin by recalling the definitions of a prime number and a composite number.

Definition 2.7. A natural number $n > 1$ is called a **prime number** if it has exactly two natural divisors: 1 and n.

The set of prime numbers is denoted by $\mathbb{P}$. At the time of writing this text, the largest known prime number is

$$2^{136\ 279\ 841} - 1,$$

which has over 41 million digits and was discovered on October 12, 2024. The following theorem was already known to the ancient Greeks.

Theorem 2.8. *The set $\mathbb{P}$ of prime numbers is infinite.*

Definition 2.9. A natural number $n > 1$ is called a **composite number** if it is not a prime number.

Note that there are also infinitely many composite numbers. It suffices to consider all powers of any natural number greater than 1.

Remark 2.10. The number 1 is neither a prime number nor a composite number.

The aim of this subsection is to present theorems regarding the uniqueness of the factorization of natural and integer numbers, which we present below.

Theorem 2.11. *Every natural number $n > 1$ can be uniquely expressed as a product of powers of prime numbers:*

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$.

Theorem 2.12. *Every integer n different from $0, 1, -1$ can be uniquely expressed in the form:*

$$n = \operatorname{sgn}(n) \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$.

The function sgn used in the above theorem guarantees that we account for the sign of the number n, and its values are as follows:

$$\operatorname{sgn}(n) = \begin{cases} -1, & \text{if } n < 0, \\ 0, & \text{if } n = 0, \\ 1, & \text{if } n > 0. \end{cases}$$

The factorizations occurring in the above theorems are called **canonical factorizations** or **prime factorizations**.

Example 2.13. Let us recall the method for finding the canonical factorization. We start by writing the number for which we are seeking the factorization and a vertical line to its right. In each step of the algorithm, for the number n—the last (written lowest) number on the left side of the line—we look for the smallest prime number p that divides the number n. We write the number p to the right of the line opposite the number n. Below n, on the left side, we write the quotient n/p. This step is repeated until we obtain a quotient equal to 1. The operation of this algorithm is illustrated below using the example of the number $a = 1260$:

$$
\begin{array}{r|l} 1260 & \\ & \\ & \\ & \\ & \\ & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & \\ & \\ & \\ & \\ & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & 2 \\ 315 & \\ & \\ & \\ & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & 2 \\ 315 & 3 \\ 105 & \\ & \\ & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & 2 \\ 315 & 3 \\ 105 & 3 \\ 35 & \\ & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & 2 \\ 315 & 3 \\ 105 & 3 \\ 35 & \\ 7 & \\ & \end{array}
\qquad
\begin{array}{r|l} 1260 & 2 \\ 630 & 2 \\ 315 & 3 \\ 105 & 3 \\ 35 & 5 \\ 7 & 7 \\ 1 & \end{array}
$$

The obtained canonical factorization has the form $a = 2^2 \cdot 3^2 \cdot 5 \cdot 7$. Note that in subsection 2.1 (p. 38), we presented divisibility rules that we can apply here to quickly check whether the successive quotients are divisible by small prime numbers: $2, 3, 5, \ldots$.

In the case of a negative number, the procedure is the same except for the final answer, where we need to account for the sign of the number, e.g., $-1260 = (-1) \cdot 2^2 \cdot 3^2 \cdot 5 \cdot 7$.

Remark 2.14. Based on the canonical factorization of the number n in example 2.13 (p. 40), we can determine the number of its divisors. Note that every natural divisor of the number n has the form $2^x \cdot 3^y \cdot 5^z \cdot 7^w$, where $x \in \{0, 1, 2\}$, $y \in \{0, 1, 2\}$, $z \in \{0, 1\}$, and $w \in \{0, 1\}$. By multiplying the number of possible values that each exponent can take, we obtain $3 \cdot 3 \cdot 2 \cdot 2 = 36$. Thus, the number of natural divisors is 36, while the number of integer divisors is $2 \cdot 36 = 72$. We encourage the reader to carry out similar reasoning for smaller numbers and to derive a general formula for the number of divisors for a number that has a specified canonical factorization. It is worth comparing this remark with Exercises 3.64 (p. 70) and 3.127 (p. 75).

2.3. Greatest Common Divisor and Least Common Multiple

In this subsection, we will demonstrate how to use the canonical factorization of numbers to determine their greatest common divisor and least common multiple. Let us recall the definitions of these concepts.

Definition 2.15. The **greatest common divisor** of numbers $a_1, a_2, \ldots, a_k \in \mathbb{Z}$, where $k \in \mathbb{N}$ and not all these numbers are zero, denoted by $GCD(a_1, a_2, \ldots, a_k)$, is the largest natural number that divides each of these numbers.

Definition 2.16. The **least common multiple** of numbers $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{N}$, denoted by $LCM(a_1, a_2, \ldots, a_k)$, is the smallest natural number that is divisible by each of these numbers.

Example 2.17. Let $a = 23\,760$, $b = 69\,300$. We factor the given numbers into prime factors as we demonstrated in Example 2.13 (p. 40), obtaining:

$$a = 2^4 \cdot 3^3 \cdot 5 \cdot 11, \quad b = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11.$$

Then:

1. $GCD(a, b) = 2^2 \cdot 3^2 \cdot 5 \cdot 11 = 1980$.
   We select the prime factors that appear in both factorizations: $2, 3, 5, 11$. For each prime factor, we take the **smaller** exponent from both factorizations: 2^2, 3^2, 5^1, 11^1. We multiply the resulting factors.

2. $LCM(a, b) = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 = 831\,600$.
   We select all prime factors that appear in at least one of the factorizations: $2, 3, 5, 7, 11$. For each prime factor, we take the **larger** exponent from both factorizations: 2^4, 3^3, 5^2, 7^1, 11^1. We multiply the resulting factors.

Now we will list some basic properties of the greatest common divisor and least common multiple. Let $a, d, m, a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{0\}$, where $k \in \mathbb{N}$. Then:

1. If $d \mid a_1$, $d \mid a_2$, …, $d \mid a_k$, then $d \mid GCD(a_1, a_2, \ldots, a_k)$,
2. If $a_1 \mid m$, $a_2 \mid m$, …, $a_k \mid m$, then $LCM(a_1, a_2, \ldots, a_k) \mid m$,
3. $GCD(a_1, a_2) \cdot LCM(a_1, a_2) = |a_1 \cdot a_2|$,
4. $GCD(a_1, \ldots, a_{k-2}, a_{k-1}, a_k) = GCD(a_1, \ldots, a_{k-2}, GCD(a_{k-1}, a_k))$,
5. $LCM(a_1, \ldots, a_{k-2}, a_{k-1}, a_k) = LCM(a_1, \ldots, a_{k-2}, LCM(a_{k-1}, a_k))$,
6. $GCD(a, 1) = 1$, $GCD(a, -1) = 1$,
7. If $a_1 \mid a$, $a_2 \mid a$ and $GCD(a_1, a_2) = 1$, then $a_1 a_2 \mid a$.

In subsection 2.4, in Examples 2.22, 2.23, 2.24, and 2.25 (p. 44-45), we will demonstrate how to apply the above properties in solving problems.

While listing divisibility rules on page 38, we omitted the numbers 6, 10, and 12. These are composite numbers whose canonical factorizations contain more than one prime factor. In such cases, we use divisibility rules for prime numbers or their powers along with property 7 presented above. From this arise the following divisibility rules:

- $6 \mid n$ if both $2 \mid n$ and $3 \mid n$.
- $10 \mid n$ if both $2 \mid n$ and $5 \mid n$, which means that the last digit of the number n is 0.
- $12 \mid n$ if both $3 \mid n$ and $4 \mid n$.
- $14 \mid n$ if both $2 \mid n$ and $7 \mid n$.
- $15 \mid n$ if both $3 \mid n$ and $5 \mid n$.

Example 2.18.

1. Let $e = 27\,526\,158$. Then $6 \mid e$, because the last digit of e is 8, which is even, and also because $2+7+5+2+6+1+5+8 = 36$ and $3 \mid 36$ (we checked divisibility by 2 and 3).
2. Let $i = 8\,651\,370$. Then $10 \mid i$, because the last digit of i is 0.
3. Let $k = 6\,507\,720$. Then $12 \mid k$, because $6+5+0+7+7+2+0 = 27$ and $3 \mid 27$, while at the end of k we have the two-digit number 20, which is divisible by 4 (we checked divisibility by 3 and 4).
4. Let $m = 1\,255\,128$. Then $14 \mid m$ because the last digit of m is 8, which is even, and $1 - 255 + 128 = -126$ and $7 \mid -126$ (we checked divisibility by 2 and 7).
5. Let $n = 978\,345$. Then $15 \mid n$, because $9+7+8+3+4+5 = 36$ and $3 \mid 36$, and the last digit of n is 5 (we checked divisibility by 3 and 5).

2.4. Euclidean Algorithm

The Euclidean algorithm presents a method for determining the greatest common divisor. It is one of the oldest known algorithms: its first description appeared in the work of Euclid entitled "Elements" around three hundred years before our era. Its simplicity and efficiency make it a widely known and commonly used algorithm to this day.

Algorithm 2.19. (Euclidean Algorithm)

Input: $a, b \in \mathbb{Z}$, $|b| < |a|$.
Output: $GCD(a, b)$.

1. We define the sequences $r_{-1}, r_0, r_1, \ldots$ and $q_1, q_2, \ldots$ as follows:
   (a) We set $r_{-1} = a$, $r_0 = b$;
   (b) If we have already determined the numbers $r_{-1}, r_0, r_1, \ldots, r_{k-1}$, then r_k and q_k are calculated from the equality (2.1, p. 38), that is:
   $$r_{k-2} = q_k r_{k-1} + r_k, \qquad 0 \leq r_k < r_{k-1}; \tag{2.2}$$
   (c) We continue determining the sequences $r_{-1}, r_0, r_1, \ldots$ and $q_1, q_2, \ldots$ until we have $r_n = 0$ for some $n \in \mathbb{N}$.
2. We return the result $GCD(a, b) = r_{n-1}$.

Note that the quotients calculated during the application of the above algorithm do not affect the value of the greatest common divisor. Therefore, in practice, we omit them, determining only the remainders in subsequent steps and using the equality:

$$GCD(qb + r, b) = GCD(b, r), \quad q, b, r \in \mathbb{Z},\ 0 \leq r < b. \tag{2.3}$$

Example 2.20. Let $a = 111$, $b = 48$. To carry out the Euclidean algorithm, we propose constructing a convenient table, which we present below. In the first row, we write the equality (2.2) for $r_{-1} = a = 111$, $r_0 = b = 48$, determining $q_1 = 2$ and $r_1 = 15$. In the second row, for $r_0 = 48$, $r_1 = 15$, we determine $q_2 = 3$ and $r_2 = 3$. The remainder is still non-zero, so we continue the algorithm: in the third row, for $r_1 = 15$, $r_2 = 3$, we determine $q_3 = 5$ and $r_3 = 0$. Since the remainder r_3 is equal to zero, we conclude the algorithm by taking $GCD(48, 111) = r_2 = 3$ as the last non-zero remainder.

k	$r_{k-2} = q_k \cdot r_{k-1} + r_k$
1	$111 = 2 \cdot 48 + 15$
2	$48 = 3 \cdot 15 + 3$
3	$15 = 5 \cdot 3 + 0$

Using the formula (2.3), we can express the operation of the Euclidean algorithm as a sequence of equalities:

$$GCD(111, 48) = GCD(48, 15) = GCD(15, 3) = GCD(3, 0) = 3.$$

Example 2.21. The presented method can be applied to three or more numbers. Suppose we want to determine $GCD(105, 147, 161)$. In successive steps, we write down the smallest non-zero number and replace the other numbers with the remainders from dividing by it. We continue this procedure until we have all remainders—except one—equal to zero:

$GCD(105, 147, 161) = GCD(105, 42, 56)$, because the remainder of dividing 147 by 105 is 42 and the remainder of dividing 161 by 105 is 56. We continue:

$GCD(105, 42, 56) = GCD(42, 21, 14)$, because the remainder of dividing 105 by 42 is 21 and the remainder of dividing 56 by 42 is 14. Next:

$GCD(42, 21, 14) = GCD(14, 0, 7)$, because the remainder of dividing 42 by 14 is 0 and the remainder of dividing 21 by 14 is 7. We continue:

$GCD(14, 0, 7) = GCD(7, 0, 0)$, because the remainder of dividing 14 by 7 is 0 and the remainder of dividing 0 by 7 is also 0. We conclude the operation of the algorithm:

$GCD(7, 0, 0) = 7$.

Let us summarize the methods we have learned for determining the greatest common divisor and least common multiple. To find the greatest common divisor, we have at our disposal:

1. prime factorization (Example 2.17, p. 41), a method for two or more numbers,
2. the Euclidean algorithm (full notation, Example 2.20, p. 43), a method for exactly two numbers,
3. the Euclidean algorithm (quick notation, Example 2.21, p. 43), a method for two or more numbers.

To determine the least common multiple, we currently have prime factorization (Example 2.17, p. 41) available—this is a method for two or more numbers. Note that in subsection 2.3, we have property 3 (p. 42), which can be used to find the least common multiple of two numbers, provided that we know their greatest common divisor. A relevant example is given below.

Example 2.22. Let $a = 111$, $b = 48$. Then based on Example 2.20 (p. 43), we have $GCD(111, 48) = 3$, and using property 3 (p. 42), we obtain:

$$LCM(48, 111) = \frac{48 \cdot 111}{GCD(48, 111)} = \frac{5328}{3} = 1776.$$

The same property can be used for a certain special type of problem, as described in the following example.

Example 2.23. Let us determine the natural numbers a, b knowing that $LCM(a, b) = 63$ and $GCD(a, b) = 3$. Note that:

$$LCM(a, b) \cdot GCD(a, b) = a \cdot b = \underbrace{x \cdot GCD(a, b)}_{a} \cdot \underbrace{y \cdot GCD(a, b)}_{b},$$

for such numbers $x, y \in \mathbb{N}$ that $GCD(x, y) = 1$. Substituting the known numerical values gives us: $63 \cdot 3 = x \cdot 3 \cdot y \cdot 3$, which simplifies to the equation $x \cdot y = 21$. We are therefore looking for all pairs of natural numbers satisfying the conditions $x \cdot y = 21$ and $GCD(x, y) = 1$:

$$\begin{cases} x = 1 \\ y = 21 \end{cases} \quad \text{or} \quad \begin{cases} x = 3 \\ y = 7 \end{cases} \quad \text{or} \quad \begin{cases} x = 7 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = 21 \\ y = 1. \end{cases}$$

From this, the possible values for numbers a and b are:

$$\begin{cases} a = 3 \\ b = 63 \end{cases} \quad \text{or} \quad \begin{cases} a = 9 \\ b = 21 \end{cases} \quad \text{or} \quad \begin{cases} a = 21 \\ b = 9 \end{cases} \quad \text{or} \quad \begin{cases} a = 63 \\ b = 3. \end{cases}$$

Let us illustrate how properties 4 and 5 (p. 42) work with two simple examples.

Example 2.24. We would like to calculate $GCD(161, 147, 105)$. According to property 4 (p. 42), we can perform the calculations step by step. First, we can calculate $GCD(147, 105) = 21$, and then we replace the numbers 147 and 105 in the initial expression with this result and find $GCD(161, 21) = 7$. The appropriate notation is as follows:

$$GCD(161, 147, 105) = GCD(161, GCD(147, 105)) = GCD(161, 21) = 7.$$

Example 2.25. We would like to calculate $LCM(161, 147, 105)$. According to property 5 (p. 42), we can perform the calculations step by step. First, we can calculate $LCM(147, 105) = 735$, and then we replace the numbers 147 and 105 in the expression $LCM(161, 147, 105)$ with the obtained result and determine $LCM(161, 735) = 16905$. The appropriate notation is as follows:

$$LCM(161, 147, 105) = LCM(161, LCM(147, 105)) = LCM(161, 735) = 16905.$$

Many problems can be reduced to finding integer solutions to linear equations. In such cases, the extended version of the Euclidean algorithm becomes useful, which we will present at the end of this subsection. In addition to finding the greatest common divisor, this algorithm also determines the numbers mentioned in the statement of the following theorem.

Theorem 2.26. *Let $a, b \in \mathbb{Z}$. Then there exist integers $x, y \in \mathbb{Z}$ such that the equation $ax + by = GCD(a, b)$ holds.*

Algorithm 2.27. (Extended Euclidean Algorithm)

Input: $a, b \in \mathbb{Z}$, $|b| < |a|$.
Output: $GCD(a, b)$ and integers $x, y \in \mathbb{Z}$ such that $ax + by = GCD(a, b)$.

1. We perform the Euclidean algorithm 2.19 (p. 43), resulting in $GCD(a, b) = r_{n-1}$.

2. We define the sequences $u_{-1}, u_0, u_1, \ldots$ and $v_{-1}, v_0, v_1, \ldots$ as follows:

   (a) We set $u_{-1} = 1$, $u_0 = 0$, $v_{-1} = 0$, $v_0 = 1$;

   (b) If we have obtained the numbers $u_{-1}, u_0, u_1, \ldots, u_{k-1}$ and $v_{-1}, v_0, v_1, \ldots, v_{k-1}$, then u_k and v_k are calculated as follows:

$$u_k = u_{k-2} - q_k \cdot u_{k-1}, \qquad v_k = v_{k-2} - q_k \cdot v_{k-1}. \tag{2.4}$$

3. We return: $x = u_{n-1}$, $y = v_{n-1}$.

Example 2.28. Let $a = 357$, $b = 161$. First, we perform the Euclidean Algorithm 2.19 (p. 43), resulting in the table below and $GCD(357, 161) = 7$.

k	$r_{k-2} = q_k \cdot r_{k-1} + r_k$
1	$357 = 2 \cdot 161 + 35$
2	$161 = 4 \cdot 35 + 21$
3	$35 = 1 \cdot 21 + 14$
4	$21 = 1 \cdot 14 + 7$
5	$14 = 2 \cdot 7 + 0$

The second part of the extended Euclidean algorithm can also be carried out using a convenient table, which we construct as follows:

1. In the first column, we write the index values i, starting from -1 and ending at $n-1=4$ (here $n=5$, because the algorithm ended after five steps),
2. In the second column, we write the values of q_k for $k=1,\ldots,n-1$; these numbers are taken from the first table,
3. In the third and fourth columns, we write the initial values of the elements u_{-1}, u_0, v_{-1}, v_0, which are given in item 2a (p. 45) of the above algorithm,
4. In the third and fourth columns, we write the values of the elements u_k and v_k, which we calculate using formulas (2.4, p. 45).

k	q_k	u_k	v_k
-1	$\times$	1	0
0	$\times$	0	1
1	2	1	-2
2	4	-4	9
3	1	5	-11
4	1	-9	20

$$
\begin{aligned}
u_1 &= u_{-1} - q_1 \cdot u_0 = 1 - 2 \cdot 0 = 1\\
v_1 &= v_{-1} - q_1 \cdot v_0 = 0 - 2 \cdot 1 = -2\\
u_2 &= u_0 - q_2 \cdot u_1 = 0 - 4 \cdot 1 = -4\\
v_2 &= v_0 - q_2 \cdot v_1 = 1 - 4 \cdot (-2) = 9\\
u_3 &= u_1 - q_3 \cdot u_2 = 1 - 1 \cdot (-4) = 5\\
v_3 &= v_1 - q_3 \cdot v_2 = -2 - 1 \cdot 9 = -11\\
u_4 &= u_2 - q_4 \cdot u_3 = -4 - 1 \cdot 5 = -9 = x\\
v_4 &= v_2 - q_4 \cdot v_3 = 9 - 1 \cdot (-11) = 20 = y
\end{aligned}
$$

5. Finally, we check the correctness of the obtained result:

$$357 \cdot (-9) + 161 \cdot 20 = -3213 + 3220 = 7. \quad \checkmark$$

2.5. Relatively Prime Numbers

At the end of this chapter, we will provide definitions of relatively prime numbers, pairwise relatively prime numbers and Euler's totient function.

Definition 2.29. Let $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{N}$. We say that the numbers $a_1, a_2, \ldots, a_k$ are **relatively prime** if $GCD(a_1, a_2, \ldots, a_k) = 1$.

Definition 2.30. Let $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{N}$. We say that the numbers $a_1, a_2, \ldots, a_k$ are **pairwise relatively prime** if $GCD(a_i, a_j) = 1$ for all such pairs $i, j \in \{1, 2, \ldots, k\}$ for which $i \neq j$.

Example 2.31. Let $a = 75$, $b = 87$, $c = 121$.

1. The given numbers are relatively prime because $GCD(75, 87, 121) = 1$.
2. To check whether the given numbers are pairwise relatively prime, we need to check the greatest common divisor of each pair:

$$GCD(75, 87) = 3, \quad GCD(75, 121) = 1, \quad GCD(87, 121) = 1.$$

If the greatest common divisor of each pair were equal to 1, then these numbers would be pairwise relatively prime. However, since $GCD(75, 87) = 3$, the given numbers are not pairwise relatively prime.

Definition 2.32. The **Euler's totient function** is defined as the function $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$, which assigns to each natural number the count of numbers that are relatively prime to it and not greater than it.

The value of Euler's totient function for any natural number can be calculated using its prime factorization, as stated in the following theorem.

Theorem 2.33. *Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $k \in \mathbb{N}$, be the prime factorization of the number n. Then*

$$\varphi(n) = p_1^{\alpha_1-1}(p_1-1) \cdot p_2^{\alpha_2-1}(p_2-1) \cdots p_k^{\alpha_k-1}(p_k-1). \tag{2.5}$$

Example 2.34. Let $n = 21\,600$. First, we find the prime factorization of the number:

$$21\,600 = 2^5 \cdot 3^3 \cdot 5^2.$$

Next, using formula (2.5), we obtain:

$$\varphi(21\,600) = \varphi(2^5 \cdot 3^3 \cdot 5^2) = 2^4(2-1) \cdot 3^2(3-1) \cdot 5^1(5-1) = 5760.$$

This result means that there are 5760 numbers that are relatively prime to 21 600 and not greater than 21 600.

From Theorem 2.33, we can draw the following conclusion.

Theorem 2.35. *Euler's totient function φ is multiplicative: for relatively prime numbers $m, n \in \mathbb{N}$ we have*

$$\varphi(m \cdot n) = \varphi(m) \cdot \varphi(n).$$

2.6. Exercises

Exercise 2.1. For the divisibility properties 1 – 5 (p. 38), choose numbers $a, b, m \in \mathbb{Z}$ such that the antecedents of the implications are true. Then verify the truth of their consequents.

Exercise 2.2. Determine the remainder r and the quotient q from dividing a by b, if:

a) $a = 15$, $b = 7$,
b) $a = -15$, $b = 7$,
c) $a = 15$, $b = -7$,
d) $a = -15$, $b = -7$,
e) $a = 43$, $b = 13$,
f) $a = -43$, $b = 13$,
g) $a = 43$, $b = -13$,
h) $a = -43$, $b = -13$,
i) $a = 200$, $b = 35$,
j) $a = -200$, $b = 35$,
k) $a = 200$, $b = -35$,
l) $a = -200$, $b = -35$.

Exercise 2.3. Using appropriate divisibility rules, justify that:

a) $2 \mid 64\,536$,
b) $3 \mid 176\,895$,
c) $4 \mid 234\,968$,
d) $5 \mid 547\,855$,
e) $6 \mid 39\,504$,
f) $7 \mid 3\,948\,861$,
g) $8 \mid 417\,472$,
h) $9 \mid 856\,512$,
i) $10 \mid 145\,820$,
j) $11 \mid 2\,801\,557$,
k) $12 \mid 7\,049\,076$,
l) $13 \mid 763\,646$,
m) $14 \mid 733\,096$,
n) $15 \mid 852\,630$,
o) $18 \mid 1\,541\,214$,
p) $21 \mid 1\,180\,914$,
q) $22 \mid 1\,440\,516$,
r) $24 \mid 1\,353\,336$,
s) $26 \mid 6\,128\,148$,
t) $30 \mid 137\,010$.

Exercise 2.4. Choose any eight-digit number $x \in \mathbb{N}$. Using appropriate divisibility rules, check whether the chosen number x is divisible by the numbers from 2 to 15.

Exercise 2.5. Determine the canonical factorization of the following numbers:

a) $42\,768$, c) $17\,712$, e) $-3\,375$, g) $82\,320$,

b) $-508\,079$, d) $10\,800$, f) $6\,462\,720$, h) $-348\,480$.

Exercise 2.6. By factoring the given numbers, calculate their GCD:

a) $5,292,\ 5,544$,

b) $9,075,\ -32,175$,

c) $14,175,\ 75,600$,

d) $-3,375,\ 4,320,\ 82,320$,

e) $2,520,\ -5,184,\ 10,584$,

f) $-3,003,\ 5,040,\ -14,994$,

g) $3,024,\ 6,048,\ 9,072,\ 10,584$,

h) $3,640,\ 4,080,\ 4,320,\ -5,525$.

Exercise 2.7. Using the prime factorizations of the numbers from Exercise 2.6, find their LCM.

Exercise 2.8. Using the Euclidean algorithm (full notation), compute the GCD of the given numbers:

a) $5,\ 8$, c) $462,\ 1,260$, e) $4,370,\ 5,720$,

b) $87,\ -237$, d) $-525,\ -2,345$, f) $-6,948,\ 178,542$.

Exercise 2.9. Using the Euclidean algorithm (quick notation), compute the GCD of the numbers from Exercises 2.6 and 2.8.

Exercise 2.10. For the properties of GCD and LCM numbered 1 and 2 (p. 42), choose such numbers $d, m, a_1, a_2, a_3, a_4 \in \mathbb{Z}$ that the antecedents of the implications are true. Then verify the truth of their consequents.

Exercise 2.11. Using the appropriate formula (property 3, p. 42), calculate the LCM of the numbers from Exercise 2.8.

Exercise 2.12. Using the appropriate formula (property 4, p. 42), calculate the GCD of the given numbers:

a) 28, 42, 70,

b) 715, 990, 1001,

c) 3570, 4116, 5607,

d) 5928, 6396, 8385, 9685,

e) 8320, 9180, $11\,250$, $14\,157$,

f) $2197, 3780, 4352, 7128, 13\,310$.

Exercise 2.13. Using the appropriate formula (property 5, p. 42), calculate the LCM of the numbers from Exercise 2.12 (*Note: The resulting numbers may be very large, so it is advisable to use prime factorizations*).

Exercise 2.14. Knowing that $LCM(a, b) = 99$ and $GCD(a, b) = 11$, determine $a, b \in \mathbb{N}$.

Exercise 2.15. Knowing that $LCM(a, b) = 180$ and $GCD(a, b) = 45$, determine $a, b \in \mathbb{N}$.

Exercise 2.16. Knowing that $LCM(a, b) = 210$ and $GCD(a, b) = 3$, determine $a, b \in \mathbb{N}$.

Exercise 2.17. Knowing that $LCM(a, b) = 3120$ and $GCD(a, b) = 13$, determine $a, b \in \mathbb{N}$.

Exercise 2.18. Knowing that $LCM(a, b) = 1980$ and $GCD(a, b) = 11$, determine $a, b \in \mathbb{N}$.

Exercise 2.19. Using the extended Euclidean algorithm, compute $GCD(a, b)$ and such integers $x, y \in \mathbb{Z}$ that $ax + by = GCD(a, b)$, if:

a) $a = 61$, $b = 7$,

b) $a = -84$, $b = 15$,

c) $a = 123$, $b = 93$,

d) $a = 241$, $b = -79$,

e) $a = 377$, $b = 123$,

f) $a = -533$, $b = -187$,

g) $a = 777$, $b = 555$,

h) $a = -1140$, $b = 570$,

i) $a = 76\,501$, $b = 29\,719$.

Exercise 2.20. Check whether the following numbers are relatively prime and whether they are pairwise relatively prime:

a) 24, 33, 44,

b) 150, 169, 325,

c) 196, 225, 289,

d) 256, 729, 3125,

e) 1918, 4080, 5125, 6375,

f) 4913, 5733, 6859, 14 641.

Exercise 2.21. Determine the value of Euler's function φ for the numbers from Exercise 2.5. (*Note: If a given number is negative, consider its opposite number.*)

2.7. Answers

Answer 2.1. —

Answer 2.2.

a) $q = 2$, $r = 1$,

b) $q = -3$, $r = 6$,

c) $q = -2$, $r = 1$,

d) $q = 3$, $r = 6$,

e) $q = 3$, $r = 4$,

f) $q = -4$, $r = 9$,

g) $q = -3$, $r = 4$,

h) $q = 4$, $r = 9$,

i) $q = 5$, $r = 25$,

j) $q = -6$, $r = 10$,

k) $q = -5$, $r = 25$,

l) $q = 6$, $r = 10$.

Answer 2.3.

a) $2 \mid 64\,536$, because the last digit of $64\,536$ is 6, which is even.

b) $3 \mid 176\,895$, because $1 + 7 + 6 + 8 + 9 + 5 = 36$ and $3 \mid 36$.

c) $4 \mid 234\,968$, because the last two digits of $234\,968$ are 68, which is divisible by 4.

d) $5 \mid 547\,855$, because the last digit of $547\,855$ is 5.

e) $6 \mid 39\,504$, because the last digit of $39\,504$ is 4, which is even, and $3 + 9 + 5 + 0 + 4 = 21$ and $3 \mid 21$.

f) $7 \mid 3\,948\,861$, because $3 - 948 + 861 = -84$ and $7 \mid 84$.

g) $8 \mid 417\,472$, because the last three digits of $417\,472$ are 472, which is divisible by 8.

h) $9 \mid 856\,512$, because $8+5+6+5+1+2=27$ and $9 \mid 27$.

i) $10 \mid 145\,820$, because the last digit of 145820 is 0.

j) $11 \mid 2\,801\,557$, because $2-8+0-1+5-5+7=0$ and $11 \mid 0$.

k) $12 \mid 7\,049\,076$, because $7+0+4+9+0+7+6=33$ and $3 \mid 33$, and the last two digits of 7049076 are divisible by 4.

l) $13 \mid 763\,646$, because $763-646=117$ and $13 \mid 117$.

m) $14 \mid 733\,096$, because the last digit of $733\,096$ is 6, which is even, and $733-096=637$ and $7 \mid 637$.

n) $15 \mid 852\,630$, because $8+5+2+6+3+0=24$ and $3 \mid 24$, and the last digit of 852630 is 0.

o) $18 \mid 1\,541\,214$, because the last digit of $1\,541\,214$ is 4, which is even, and $1+5+4+1+2+1+4=18$ and $9 \mid 18$.

p) $21 \mid 1\,180\,914$, because $1+1+8+0+9+1+4=24$ and $3 \mid 24$, and $1-180+914=735$ and $7 \mid 735$.

q) $22 \mid 1\,440\,516$, because the last digit of $1\,440\,516$ is 6, which is even, and $1-4+4-0+5-1+6=11$ and $11 \mid 11$.

r) $24 \mid 1\,353\,336$, because $1+3+5+3+3+3+6=24$ and it follows that $3 \mid 24$, and the last three digits of $1,353,336$ are 336, which is divisible by 8.

s) $26 \mid 6,128,148$, since the last digit 8 is even, and also $6-128+148=26$, thus $13 \mid 26$.

t) $30 \mid 137,010$, since $1+3+7+0+1+0=12$, hence $3 \mid 12$, while the last digit is 0.

Answer 2.4. —

Answer 2.5.

a) $2^4 \cdot 3^5 \cdot 11$,
b) $-11^2 \cdot 13 \cdot 17 \cdot 19$,
c) $2^4 \cdot 3^3 \cdot 41$,
d) $2^4 \cdot 3^3 \cdot 5^2$,
e) $-3^3 \cdot 5^3$,
f) $2^8 \cdot 3^3 \cdot 5 \cdot 11 \cdot 17$,
g) $2^4 \cdot 3 \cdot 5 \cdot 7^3$,
h) $-2^6 \cdot 3^2 \cdot 5 \cdot 11^2$.

Answer 2.6.

a) 252,
b) 825,
c) 4725,
d) 15,
e) 72,
f) 21,
g) 1512,
h) 5.

Answer 2.7.

a) $116\,424$,
b) $353\,925$,
c) $226\,800$,
d) $37\,044\,000$,
e) $1\,270\,080$,
f) $85\,765\,680$,
g) $127\,008$,
h) $33\,415\,200$.

Answer 2.8.

a) 1, b) 3, c) 42, d) 35, e) 10, f) 18.

Answer 2.9. Results as in Answers 2.6 and 2.8.

Answer 2.10. —

Answer 2.11.

a) 40, c) 13 860, e) 2 499 640,

b) 6873, d) 35 175, f) 68 917 212.

Answer 2.12.

a) 14, b) 11, c) 21, d) 13, e) 1, f) 1.

Answer 2.13.

a) 420,

b) 90 090,

c) $93\,412\,620 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 17 \cdot 89$,

d) $7\,786\,042\,680 = 2^3 \cdot 3 \cdot 5 \cdot 13 \cdot 19 \cdot 41 \cdot 43 \cdot 149$,

e) $57\,760\,560\,000 = 2^7 \cdot 3^3 \cdot 5^4 \cdot 11^2 \cdot 13 \cdot 17$,

f) $36\,078\,632\,029\,440 = 2^8 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^3 \cdot 13^3 \cdot 17$.

Answer 2.14.

$$\begin{cases} a = 11 \\ b = 99 \end{cases} \text{or} \quad \begin{cases} a = 99 \\ b = 11 \end{cases}$$

Answer 2.15.

$$\begin{cases} a = 45 \\ b = 180 \end{cases} \text{or} \quad \begin{cases} a = 180 \\ b = 45 \end{cases}$$

Answer 2.16.

$$\begin{cases} a = 3 \\ b = 210 \end{cases} \text{or} \quad \begin{cases} a = 210 \\ b = 3 \end{cases} \text{or} \quad \begin{cases} a = 6 \\ b = 105 \end{cases} \text{or} \quad \begin{cases} a = 105 \\ b = 6 \end{cases} \text{or}$$

$$\begin{cases} a = 15 \\ b = 42 \end{cases} \text{or} \quad \begin{cases} a = 42 \\ b = 15 \end{cases} \text{or} \quad \begin{cases} a = 21 \\ b = 30 \end{cases} \text{or} \quad \begin{cases} a = 30 \\ b = 21 \end{cases}$$

Answer 2.17.

$$\begin{cases} a = 13 \\ b = 3120 \end{cases} \text{or} \quad \begin{cases} a = 3120 \\ b = 13 \end{cases} \text{or} \quad \begin{cases} a = 208 \\ b = 195 \end{cases} \text{or} \quad \begin{cases} a = 195 \\ b = 208 \end{cases} \text{or}$$

$$\begin{cases} a = 39 \\ b = 1040 \end{cases} \text{or} \quad \begin{cases} a = 1040 \\ b = 39 \end{cases} \text{or} \quad \begin{cases} a = 65 \\ b = 624 \end{cases} \text{or} \quad \begin{cases} a = 624 \\ b = 65 \end{cases}$$

Answer 2.18.

$$\begin{cases} a = 11 \\ b = 1980 \end{cases} \text{or} \quad \begin{cases} a = 1980 \\ b = 11 \end{cases} \text{or} \quad \begin{cases} a = 44 \\ b = 495 \end{cases} \text{or} \quad \begin{cases} a = 495 \\ b = 44 \end{cases} \text{or}$$

$$\begin{cases} a = 55 \\ b = 396 \end{cases} \text{or} \quad \begin{cases} a = 396 \\ b = 55 \end{cases} \text{or} \quad \begin{cases} a = 99 \\ b = 220 \end{cases} \text{or} \quad \begin{cases} a = 220 \\ b = 99 \end{cases}$$

Answer 2.19.

a) $GCD(a, b) = 1$, $x = 3$, $y = -26$,

b) $GCD(a, b) = 3$, $x = -2$, $y = -11$,

c) $GCD(a, b) = 3$, $x = -3$, $y = 4$,

d) $GCD(a, b) = 1$, $x = 20$, $y = 61$,

e) $GCD(a, b) = 1$, $x = -46$, $y = 141$,

f) $GCD(a, b) = 1$, $x = -20$, $y = 57$,

g) $GCD(a, b) = 111$, $x = -2$, $y = 3$,

h) $GCD(a, b) = 570$, $x = 0$, $y = 1$,

i) $GCD(a, b) = 113$, $x = 54$, $y = -139$.

Answer 2.20.

a) Relatively prime: Y, Pairwise rel. prime: N,

b) Relatively prime: Y, Pairwise rel. prime: N,

c) Relatively prime: Y, Pairwise rel. prime: Y,

d) Relatively prime: Y, Pairwise rel. prime: Y,

e) Relatively prime: Y, Pairwise rel. prime: N,

f) Relatively prime: Y, Pairwise rel. prime: Y.

Answer 2.21.

a) 12 960,

b) 380 160,

c) 5 760,

d) 2 880,

e) 1 800,

f) 1 474 560,

g) 18 816,

h) 84 480.

Chapter 3

Elements of Combinatorics

In this chapter, we will present selected combinatorial concepts used for counting elements of finite sets. To this end, we will first introduce the definition of factorial and the binomial coefficient, and show their relationship with Pascal's triangle and Newton's binomial theorem. Next, we will discuss the concepts of variations, permutations, combinations, as well as the multiplication and addition principles. These concepts are typically covered in secondary school; however, they require thorough review, which is why they are illustrated with a large number of exercises. Finally, we will present the principle of inclusion-exclusion and Dirichlet's box principle.

3.1. Factorial and Binomial Coefficient

We will start by recalling two fundamental concepts related to combinatorics, specifically factorial and the binomial coefficient. In the second part of the subsection, we will show their relationship with Pascal's triangle and Newton's binomial theorem.

The factorial of a natural number n is the product of all natural numbers not greater than n:

$$n! = 1 \cdot 2 \cdots n.$$

Remark 3.1. During calculations, we may encounter the expression $0!$. In such cases, we accept that $0! = 1$.

Example 3.2.

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120, \qquad 10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3\,628\,800,$$

$$8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40\,320, \qquad 20! = 2\,432\,902\,008\,176\,640\,000.$$

As a fun fact, we can mention the (only) four numbers for which the sum of the factorials of their digits equals the numbers themselves:

$$1! = 1, \qquad 2! = 2, \qquad 1! + 4! + 5! = 145, \qquad 4! + 0! + 5! + 8! + 5! = 40585.$$

The Newton Symbol is a function of two arguments defined by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n, k \in \mathbb{N}_0$ and $k \leq n$. For $k > n$, we define $\binom{n}{k} = 0$. The above symbol is read as "*n choose k*".

Example 3.3. When calculating the value of the Newton symbol, it is important to remember the possibility of canceling the numerator with the denominator to avoid dealing with excessively large numbers:

1. $\displaystyle\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\cdot 2!} = \frac{3!\cdot 4\cdot 5}{3!\cdot 2!} = \frac{4\cdot 5}{2} = \frac{2\cdot 5}{1} = 2\cdot 5 = 10,$

2. $\displaystyle\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!\cdot 2!} = \frac{4!\cdot 5\cdot 6}{4!\cdot 2!} = \frac{5\cdot 6}{2} = \frac{5\cdot 3}{1} = 5\cdot 3 = 15,$

3. $$\binom{20}{15} = \frac{10!}{15!(20-15)!} = \frac{20!}{15!\cdot 5!} = \frac{15!\cdot 16\cdot 17\cdot 18\cdot 19\cdot 20}{15!\cdot 5!} = \\ = \frac{16\cdot 17\cdot 18\cdot 19\cdot 20}{2\cdot 3\cdot 4\cdot 5} = \frac{16\cdot 17\cdot 3\cdot 19}{1} = 15\,504.$$

Taking into account Remark 3.1 (p. 53), we can easily justify the following properties:

1. $\displaystyle\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1\cdot n!} = 1,$

2. $\displaystyle\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\cdot 0!} = \frac{n!}{n!\cdot 1} = 1,$

3. $\displaystyle\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{(n-1)!\cdot n}{(n-1)!} = n,$

4. $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k},$

5. $\displaystyle\binom{n}{n-1} = \binom{n}{1} = n.$

The Newton symbol is related to Pascal's triangle, a fragment of which is presented below:

$$\begin{array}{c|ccccccccccccc}
0 & & & & & & & 1 & & & & & & \\
1 & & & & & & 1 & & 1 & & & & & \\
2 & & & & & 1 & & 2 & & 1 & & & & \\
3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Pascal's triangle is an infinite triangular array of numbers, where the element in row n at position k is equal to $\binom{n}{k}$, with both rows and elements in the rows numbered starting from 0. Note that the sides of the triangle contain only the number 1, while each number inside it is the sum of the two numbers on either side of it in the row immediately above it. This observation is illustrated by the formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad 0 < k < n.$$

Example 3.4. Let us compare the following examples with Example 3.3 (p. 54):

1. The element in row 5 at position 3 is equal to $\displaystyle\binom{5}{3} = 10.$
2. The element in row 6 at position 4 is equal to $\displaystyle\binom{6}{4} = 15.$

At the end of this subsection, we will present the **Newton binomial**, which is a formula describing the terms of the expansion of powers of the sum of two numbers, valid for any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\begin{aligned}
(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \\
&= \binom{n}{0} a^{n-0} b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \\
&\qquad\qquad + \binom{n}{n-1} a^{n-(n-1)} b^{n-1} + \binom{n}{n} a^{n-n} b^n = \\
&= a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + n a b^{n-1} + b^n.
\end{aligned}$$

Example 3.5. Using the above formula, we can expand the following expressions:

1. $(a+b)^2 = \displaystyle\binom{2}{0} a^{2-0} b^0 + \binom{2}{1} a^{2-1} b^1 + \binom{2}{2} a^{2-2} b^2 = a^2 + 2ab + b^2.$
2. $(a+b)^3 = \displaystyle\binom{3}{0} a^{3-0} b^0 + \binom{3}{1} a^{3-1} b^1 + \binom{3}{2} a^{3-2} b^2 + \binom{3}{3} a^{3-3} b^3 =$

   $= a^3 + 3a^2 b + 3ab^2 + b^3.$
3. $(a+b)^4 = \displaystyle\binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 =$

   $= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4.$

Note that the coefficients appearing in the above expressions are consecutive elements of Pascal's triangle (p. 54) corresponding to rows 2, 3, and 4.

3.2. Variations, Permutations, and Combinations

In this subsection, we will sequentially discuss the basic concepts used for counting the elements of finite sets: variations, permutations, and combinations. Each of these concepts occurs in two variants: without repetition and with repetition. In the examples considered, the answers may be large numbers, so it is not necessary to provide final results as specific numbers; appropriate expressions will suffice.

Definition 3.6. A k-element **variation without repetition** of an n-element set A, where $k \leq n$, is defined as any k-term sequence of distinct elements, where the terms are elements of the set A.

The number of all distinct k-element variations without repetition of an n-element set, where $n, k \in \mathbb{N}$ and $k \leq n$, is equal to

$$V_n^k = \frac{n!}{(n-k)!}. \tag{3.1}$$

Example 3.7. Grandma has 5 grandchildren and bought 8 different chocolates in the store. In how many ways can Grandma give sweets to the children if each child is to receive exactly 1 chocolate?

Since Grandma is distributing chocolates, the set of chocolates will be $A = \{c_1, \ldots, c_8\}$. If we establish a certain order for the children (e.g., from the youngest to the oldest), an example of how Grandma might give chocolates to her grandchildren could be as follows: the first child receives chocolate c_3, the second receives chocolate c_1, the third receives chocolate c_8, and the fourth and fifth are given chocolates c_5 and c_7, respectively. This distribution can be concisely represented as a sequence: $(c_3, c_1, c_8, c_5, c_7)$ – it is a 5-term sequence of distinct elements, whose terms are elements of the set A. Therefore, we have $n = 8$ and $k = 5$. According to the formula (3.1), the number of such variations without repetition is

$$V_8^5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 6\,720.$$

We can arrive at the same result through the following elementary reasoning: when choosing a chocolate for the first grandchild, Grandma can choose from 8 different chocolates; for the second child, her choice decreases to 7 options, for the next two grandchildren she can choose chocolates in 6 and 5 ways respectively, and the last grandchild can receive one of the remaining 4 chocolates. By multiplying the number of possible choices at each step, we obtain the total number of variations

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6\,720.$$

Definition 3.8. A k-element **variation with repetition** of an n-element set A is defined as any k-term sequence whose terms are elements of the set A.

The number of all distinct k-element variations with repetition of an n-element set is given by

$$\overline{V}_n^k = n^k, \quad \text{where } n, k \in \mathbb{N}. \tag{3.2}$$

Example 3.9. Alicja has a test consisting of 9 questions to solve. For each question, she can select one of 3 answers (a, b, or c), or she can leave the question unanswered (un). In how many ways can Alicja complete the test?

For each question, the set of possible choices for Alicja is the same: it will be our set $A = \{a, b, c, un\}$. For example, suppose Alicja answered questions 3, 5, and 8 with a, answered questions 1 and 7 with b, answered questions 4 and 9 with c, and left questions 2 and 6 unanswered. This completed test can be described by the variation: $(b, un, a, c, a, un, b, a, c)$ – a 9-term sequence whose terms are elements of the set A. Thus, we have $n = 4$ and $k = 9$, and the number of such variations with repetition, according to the formula (3.2, p. 56), is given by

$$\overline{V}_4^9 = 4^9 = 262\,144.$$

Another way to solve this problem is to notice that for each of the nine questions, there are 4 ways to provide an answer, and then to multiply the number of possibilities for each question:

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^9 = 262\,144.$$

Definition 3.10. A **permutation without repetition** of an n-element set A is defined as any n-term sequence formed from all the elements of that set, meaning any arrangement of its elements.

The number of all distinct permutations without repetition of an n-element set is given by

$$P_n = n!, \quad \text{where } n \in \mathbb{N}. \tag{3.3}$$

Example 3.11. An art exhibition is being prepared in the gallery. On a certain wall, 8 selected paintings by Picasso need to be hung in a row. In how many ways can this be done?

In this problem, the set A is the set of paintings: $A = \{o_1, o_2, \ldots, o_8\}$. By hanging the paintings in order from left to right: $o_5, o_8, o_2, o_7, o_1, o_6, o_3, o_4$, we obtain an example of a permutation without repetition: $(o_5, o_8, o_2, o_7, o_1, o_6, o_3, o_4)$ — an 8-term sequence whose terms are all the elements of the set A. Since we have $n = 8$, the number of permutations without repetition is given by the formula (3.3):

$$P_8 = 8! = 40\,320.$$

We can also look at this problem in a slightly different way. It is enough to notice that for the first position on the wall we can choose any of the 8 paintings; then for the second position we can choose one of the remaining 7 paintings; for the third position we can place one of the 6 paintings not yet chosen; …; for the second to last position we can choose one of two paintings; and for the last position we hang the final painting. By multiplying the possibilities at each step we obtain the total number of permutations:

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8! = 40\,320.$$

Definition 3.12. A n-element **permutation with repetitions** of a k-element set $A = \{a_1, a_2, \ldots, a_k\}$, in which element a_1 appears n_1 times, element a_2 appears n_2 times, $\ldots$, element a_k appears n_k times, where $n_1 + n_2 + \cdots + n_k = n$, is defined as any n-term sequence in which element a_i appears n_i times for $i \in \{1, \ldots, k\}$.

The numbers n_i appearing in the above definition are called the multiplicities of the elements a_i in the permutation with repetitions. The number of all distinct n-element permutations with repetitions having multiplicities $n_1, n_2, \ldots, n_k \in \mathbb{N}$, such that $n_1 + n_2 + \cdots + n_k = n$, is given by

$$P_n^{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}. \tag{3.4}$$

Example 3.13. In how many ways can 4 orange juices (o), 5 apple juices (a), 3 pineapple juices (p), and 6 grapefruit juices (g) be arranged in a row on a shelf, assuming that juices of the same flavor are indistinguishable?

A sample arrangement of the juices could be as follows: the orange juice is in positions (counting from the left) 7, 8, 13, and 15; the apple juice is in positions 1, 5, 9, 14, and 17; the pineapple juice is in positions 2, 4, and 10; while the grapefruit juice is in positions 3, 6, 11, 12, 16, and 18. We can represent this arrangement as a sequence $(a, p, g, p, a, g, o, o, a, p, g, g, o, a, o, g, a, g)$, whose elements are from the set $A = \{o, a, p, g\}$ of juice types. This sequence has 18 elements, and the elements o, a, p, and g appear in this sequence 4, 5, 3, and 6 times respectively. The number of such permutations with repetitions is given by the formula (3.4):

$$P_{18}^{4,5,3,6} = \frac{18!}{4! \cdot 5! \cdot 3! \cdot 6!} = 514\,594\,080.$$

Definition 3.14. A k-element **combination without repetition** of an n-element set A, where $k \leq n$, is defined as any k-element subset of that set.

The number of all distinct k-element combinations without repetition of an n-element set is given by

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{where } k \leq n,\ n, k \in \mathbb{N}. \tag{3.5}$$

Example 3.15. At a party, there were 14 acquaintances. How many handshakes occurred if each person greeted every other person?

Let us denote the set of acquaintances as $A = \{z_1, z_2, \ldots, z_{14}\}$. Each handshake involves a pair of acquaintances: for example, if persons z_6 and z_{11} shook hands, we have a two-element combination without repetition of the form $\{z_6, z_{11}\}$. Thus, we have $n = 14$ and $k = 2$, and therefore the formula (3.5) tells us that the number of such combinations without repetition is given by

$$C_{14}^2 = \binom{14}{2} = \frac{14!}{2!(14-2)!} = \frac{14!}{2! \cdot 12!} = \frac{13 \cdot 14}{2} = 91.$$

Before the next definition, we will informally introduce the concept of a multiset, which can be regarded as a generalization of the concept of a set. In multisets, as in sequences, elements can occur more than once, and two multisets are considered equal if the multiplicities of the same elements are equal. However, unlike sequences, and similarly to sets, the order in which we list the elements of a multiset is not important. For example, the multisets $\{a, a, b, c\}$ and $\{b, a, c, a\}$ are considered identical because they contain the same elements with the same multiplicities, while the multiset $\{a, b, c\}$ will be different from them because the multiplicity of element a in this set is different.

Definition 3.16. A k-element **combination with repetitions** of an n-element set $A = \{a_1, a_2, \ldots, a_n\}$ is defined as any sequence $(k_1, k_2, \ldots, k_n)$ such that $k_1 + \cdots + k_n = k$, where $k_i \in \mathbb{N}_0$ for $i \in \{1, \ldots, n\}$.

The combination with repetitions defined above can be interpreted as a multiset containing k elements, in which element a_i appears with multiplicity k_i, for $i \in \{1, \ldots, n\}$. The number of all distinct k-element combinations with repetitions of an n-element set is given by

$$\overline{C}_n^k = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}, \quad \text{where } n, k \in \mathbb{N}. \tag{3.6}$$

Example 3.17. How many different sets containing 7 balloons can be formed, having an unlimited number of red (r), green (g), and blue (b) balloons? We assume that balloons of the same color are indistinguishable.

Let us denote the set of balloon colors as $A = \{r, g, b\}$. An example of a set may contain 2 red balloons, 3 green balloons, and 2 blue balloons, which gives us a 7-element combination with repetitions of the set of balloon colors in the form $(2, 3, 2)$, meaning a 3-term sequence where $2+3+2 = 7$. Another example could be the combination $(4, 0, 3)$, which is a 3-term sequence where $4 + 0 + 3 = 7$, interpreted as a set where there are 4 red balloons, 3 blue balloons, and no green balloons.

In this problem, we have $k = 7$ and $n = 3$, so the formula (3.6) tells us that the number of such combinations with repetitions is given by

$$\overline{C}_3^7 = \binom{3+7-1}{3-1} = \binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9!}{2! \cdot 7!} = \frac{8 \cdot 9}{2} = 36.$$

3.3. The Multiplication and Addition Principles

In this subsection, we will discuss two additional tools for counting the elements of finite sets: the multiplication principle and the addition principle. Recall that we defined the concept of a Cartesian product in subsection 1.3 (Definition 1.19, p. 16).

Theorem 3.18. (Multiplication Principle) *If $A_1, A_2, \ldots, A_n$ are finite sets, then*

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|. \tag{3.7}$$

Example 3.19. If $A_1 = \{1, 2, 3\}$, $A_2 = \{a, b\}$, $A_3 = \{2, 3, b, c\}$, then

$$|A_1 \times A_2 \times A_3| = |A_1| \cdot |A_2| \cdot |A_3| = 3 \cdot 2 \cdot 4 = 24.$$

Example 3.20. Emil decided to have lunch at the bar consisting of soup, a main course, a salad, and a dessert. In how many ways can Emil compose his meal if he has 2 different soups, 10 different main courses, 7 different salads, and 3 different desserts to choose from?

In this problem, the set A_1 is the set of soups, A_2 is the set of main courses, A_3 is the set of salads, and A_4 is the set of desserts, with $n = 4$. We denote the set of soups as $A_1 = \{s_1, s_2\}$, the set of main courses as $A_2 = \{m_1, m_2, \ldots, m_{10}\}$, the set of salads as $A_3 = \{l_1, l_2, \ldots, l_7\}$, and the set of desserts as $A_4 = \{d_1, d_2, d_3\}$. An example of Emil's lunch can be described as (s_2, m_5, l_4, d_3), which means Emil chose soup number 2, main course number 5, salad number 4, and dessert number 3. Using the formula (3.7, p. 59), we can calculate the total number of different possible lunches for Emil:

$$|A_1 \times A_2 \times A_3 \times A_4| = |A_1| \cdot |A_2| \cdot |A_3| \cdot |A_4| = 2 \cdot 10 \cdot 7 \cdot 3 = 420.$$

Theorem 3.21. (Addition Principle) *If $A_1, A_2, \ldots, A_n$ are finite pairwise disjoint sets, meaning $A_i \cap A_j = \varnothing$ for $i \neq j$, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|. \tag{3.8}$$

Example 3.22. For the sets $A_1 = \{1, 2, 3\}$, $A_2 = \{5, 7\}$, and $A_3 = \{4, 9\}$, we have $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \varnothing$, so we can apply the addition principle, which gives us

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 3 + 2 + 2 = 7.$$

Example 3.23. How many committees consisting of 4 people can be formed from a group of 9, if two people in this group, Agnieszka and Bogdan, do not want to be in the same committee?

Let us note that each committee is a subset of the given group of people. Therefore, the number of all four-person committees is equal to the number of 4-element combinations without repetition from a 9-element set, which is represented as $\binom{9}{4}$ (Definition 3.14 and formula (3.5), p. 58). However, we are interested in the number of committees that meet an additional condition: Agnieszka and Bogdan cannot be selected together. We can distinguish three subsets among the committees that satisfy this condition:

- A_1 — the set of committees that include Agnieszka but do not include Bogdan,
- A_2 — the set of committees that include Bogdan but do not include Agnieszka,
- A_3 — the set of committees that include neither Agnieszka nor Bogdan.

These sets are pairwise disjoint because they refer to mutually exclusive situations. Let us observe that:

$|A_1| = \binom{7}{3}$, because we choose 3 people (Agnieszka is already included in the considered committees) from 7 people (excluding both Agnieszka and Bogdan),

$|A_2| = \binom{7}{3}$, because we choose 3 people (Bogdan is already included in the considered committees) from 7 people (excluding both Agnieszka and Bogdan),

$|A_3| = \binom{7}{4}$, because we choose 4 out of 7 people for each committee (excluding both Agnieszka and Bogdan).

Thus, applying formula (3.8), the total number of possible 4-person committees selected from a group of 9 people, to which neither of the two specified individuals belongs simultaneously, is given by

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 105.$$

3.4. Various Problems

When solving combinatorial problems, one can encounter many situations that do not fit neatly into the schemes presented in subsections 3.2 and 3.3. Usually, solving such problems requires some experience (hence the many exercises in subsection 3.6) and skill in selecting the appropriate tools. In this subsection, we will present several practical tips through examples that may help in solving combinatorial problems, which are not necessarily standard.

To begin with, let us emphasize that a fundamental issue is a proper understanding of the problem, as often the solution does not require any calculations, and the answer is immediate!

Example 3.24. In how many different ways can 6 different greeting cards be placed into 10 identical envelopes? We can put at most 1 card into each envelope, and some envelopes may remain empty.

Since all envelopes are identical, every arrangement of cards in envelopes will be indistinguishable from others, so the answer is: there is only 1 way.

Example 3.25. In how many ways can 9 identical pictures be placed into 8 identical frames? Each frame can contain at most 1 picture, and no picture can be left without a frame.

Since there are more pictures than frames, it is impossible to arrange the pictures in the frames such that each frame contains at most 1 picture. Therefore, the answer is: 0 ways.

In the next two examples, we will analyze four situations where order plays an important role.

Example 3.26. In a class, there are 27 students: 15 girls and 12 boys. For St. Nicholas Day, the class organized a lottery with 27 tickets, among which there are exactly 5 winning tickets. In how many ways can we choose 3 girls and 2 boys who won prizes if:

a) the prizes are identical?

   Let A_1 be the family of 3-element subsets of the set consisting of 15 girls, and let A_2 be the family of 2-element subsets chosen from the set of 12 boys. The elements of sets A_1 and A_2 are combinations without repetition, so we have $|A_1| = \binom{15}{3}$ and $|A_2| = \binom{12}{2}$ (Definition 3.14 and formula (3.5), p. 58). To obtain the final answer, note that any group of five students receiving prizes can be obtained by combining any trio of girls from set A_1 with any pair of boys from set A_2. Therefore, we are dealing with a case of the multiplication principle (Theorem 3.18, p. 59), which means that the final answer is: $|A_1 \times A_2| = |A_1| \cdot |A_2| = \binom{15}{3} \cdot \binom{12}{2}$.

b) the prizes are distinct?

   According to the previous point, we can choose the group of five children who will receive prizes in $\binom{15}{3} \cdot \binom{12}{2}$ ways. Assuming that the prizes are numbered from 1 to 5, the order among the children with winning tickets must also be established, such that the first student receives the first prize, the second wins prize number 2, and so on. Arranging a set is a permutation without repetition (Definition 3.10 and formula (3.3), p. 57), and thus we can do this in 5! ways. Therefore, we obtain the final answer: $\binom{15}{3} \cdot \binom{12}{2} \cdot 5!$.

Example 3.27. The class consists of 20 students. The teacher decided to organize the work into 5 groups of 4 students each. In how many ways can the division be made if:

a) the groups are numbered from 1 to 5?

In this problem, we are dealing with determining 5 combinations without repetition (Definition 3.14 and formula (3.5), p. 58) as successive groups. Group number 1 can be chosen in $\binom{20}{4}$ ways, group number 2 can be chosen in $\binom{16}{4}$ ways, group number 3 can be chosen in $\binom{12}{4}$ ways, group number 4 can be chosen in $\binom{8}{4}$ ways, and group number 5 can be chosen in $\binom{4}{4}$ ways. Therefore, the answer is $\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}$.

b) the groups are not numbered?

According to the above, the number of numbered groups is $\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}$. Note that for each division of the class into 5 unnumbered groups of 4 students, we can assign numbers to these groups, that is, we can order the groups in exactly 5! ways (permutation without repetition, Definition 3.10 and formula 3.3, p. 57). Thus, after removing the numbering, these 5! ways give us the same division of the class into unnumbered groups. Therefore, the answer is $\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}/5!$.

Let us compare this example with Example 3.26 (p. 61). In subsection 3.26a, we select two subsets: a subset of 3 girls and a subset of 2 boys, but these are subsets from two different sets – the set of 15 girls and the set of 12 boys. In subsection 3.27a, we select successive 4-member groups, but these are subsets of the same set of 20 students in the class; therefore, in subsequent steps, when selecting members for groups 2, 3, 4, and 5, the number of students from which we choose decreases. In such tasks, it is always important to pay special attention to the sets from which we are selecting subsets.

Note that in this example alternative solutions are possible. Thus, in subsection a), we can recognize a permutation with repetitions: each of the $n = 20$ students is assigned to one of $k = 5$ groups, with the requirement that each group number appears 4 times: $n_1 = n_2 = n_3 = n_4 = n_5 = 4$, which gives us the result (Definition 3.12, formula 3.4, p. 58): $20!/(4!4!4!4!4!)$. For subsection b), we can carry out the following construction. The first student must belong to some group. From among the remaining 19 students, we choose 3 additional members for that group in $\binom{19}{3}$ ways. The first of the remaining students belongs to another group, and from among the 15 students not yet considered, we choose 3 members in $\binom{15}{3}$ ways. Continuing this process for subsequent groups, we have $\binom{11}{3}$, $\binom{7}{3}$, and $\binom{3}{3} = 1$ ways to choose. Ultimately, we can make divisions into unnumbered groups in $\binom{19}{3} \cdot \binom{15}{3} \cdot \binom{11}{3} \cdot \binom{7}{3}$ ways. We encourage readers to verify (in any way) that these results are equal to those obtained earlier.

The next four examples illustrate how important the order of drawing and whether the drawn objects are distinguishable are.

Example 3.28. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are **indistinguishable**, and the order of drawing is **important**.

In this problem, we are dealing with a case of permutations with repetitions (Definition 3.12 and formula (3.4), p. 58). The set $A = \{b, c\}$ is the set of colors, $k = 2$, $n = 6$, $n_1 = 2$, $n_2 = 4$. Note that the numbers 9 and 7 do not affect the answer here because we are drawing 2 white balls from 9 identical ones and 4 black balls from 7 identical ones. Therefore, the answer is $\frac{6!}{2! \cdot 4!}$.

Example 3.29. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are **indistinguishable**, and the order of drawing is **not important**.

We want to choose 2 white balls from the 9 available. Since the balls are indistinguishable, there is only 1 way to do this. Similarly, for the black balls, since they are indistinguishable, there is only 1 way to choose 4 balls from the 7 available. Since the order of drawing is also not important, there is only 1 way to draw such a set of balls.

Example 3.30. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are **distinguishable**, and the order of drawing is **not important**.

This task is analogous to Example 3.26a (p. 61), so the answer is $\binom{9}{2} \cdot \binom{7}{4}$.

Example 3.31. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are **distinguishable**, and the order of drawing is **important**.

This task is analogous to Example 3.26b (p. 61), so the answer is $\binom{9}{2} \cdot \binom{7}{4} \cdot 6!$.

At the end of this subsection, we will show through examples that in some problems it is worthwhile to first consider the opposite condition to that described in the problem.

Example 3.32. In a certain group of 10 people, there are Cezary and Daria. How many ways can these people be arranged in a line so that Cezary and Daria do not stand next to each other?

The arrangement of the set is a permutation without repetition (Definition 3.10 and formula (3.3), p. 57), so the number of ways to arrange 10 people in a line is $10!$. Let's consider how many possible situations exist where Cezary and Daria stand next to each other. We can assume that Cezary and Daria are holding hands, and we permute this pair with the remaining 8 people, which gives us $9!$ possibilities. Additionally, we can arrange Cezary and Daria within their pair in 2 ways. Therefore, there are $2 \cdot 9!$ possibilities to arrange the 10 people in a line such that Cezary and Daria are next to each other. If we are interested in the opposite situation, where Cezary and Daria do not stand next to each other, we simply subtract the obtained result from all possible arrangements of the entire group in a line, which is $(10! - 2 \cdot 9!$.

Example 3.33. We roll three six-sided dice: yellow, red, and green. How many possible situations are there in which the product of the number of spots rolled on the dice is divisible by 3?

The result of rolling three different dice can be treated as a variation with repetition (Definition 3.8 and formula (3.2), p. 56). Here, the set $A = \{1, 2, 3, 4, 5, 6\}$ represents the possible outcomes of rolling one die. Since $n = 6$ and $k = 3$, the total number of possibilities is 6^3. The product is divisible by 3 if at least one factor of the product is divisible by 3. Let's consider the opposite condition: for the product of the numbers on the three dice to not be divisible by 3, the result on each die cannot be divisible by 3, meaning it must belong to the set $A' = \{1, 2, 4, 5\}$. Therefore, the total number of rolls of three dice for which the product of the numbers is not divisible by 3 is 4^3 (again a variation with repetition, where $n' = |A'| = 4$ and $k' = 3$). To find the number of possibilities when the product of the numbers is divisible by 3, we simply subtract the obtained result from the total number of possible outcomes when rolling 3 dice: $6^3 - 4^3$.

3.5. Inclusion-Exclusion Principle and Dirichlet's Box Principle

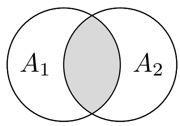
At the end of this chapter, we will present the inclusion-ixclusion principle and Dirichlet's box principle in their simplest forms. Recall that if we want to count the number of elements in the sum of pairwise disjoint sets, we use the addition principle 3.21 (p. 60). If the sets are not pairwise disjoint, we need an additional tool.

Theorem 3.34. (Inclusion-Exclusion Principle) *If $A_1, A_2, \ldots, A_n$ are finite sets, then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_n| =$$
$$= \sum_{i=1}^{n} |A_i| - \sum_{\substack{i,j=1 \\ i<j}}^{n} |A_i \cap A_j| + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|. \quad (3.9)$$

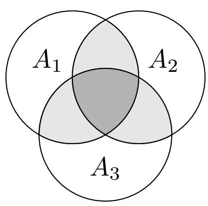
The formula (3.9) is complex. To understand what it entails, let us consider the inclusion-exclusion principle for two sets. The number of elements in the set $A_1 \cup A_2$ is usually less than $|A_1| + |A_2|$, because the elements in the intersection $A_1 \cap A_2$ (the gray area in the diagram below) are counted twice. To obtain the correct result, the term $|A_1 \cap A_2|$ must be subtracted:

$$\left|\bigcup_{i=1}^{2} A_i\right| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$



A similar reasoning can be applied for three sets. The number of elements in the set $A_1 \cup A_2 \cup A_3$ is generally less than the sum $|A_1| + |A_2| + |A_3|$, because the elements in the intersections $A_1 \cap A_2$, $A_1 \cap A_3$, and $A_2 \cap A_3$ (the gray areas in the diagram below) are counted twice. Therefore, the terms $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, and $|A_2 \cap A_3|$ must be subtracted from $|A_1| + |A_2| + |A_3|$. However, this way, the elements in the intersection $A_1 \cap A_2 \cap A_3$ (the dark gray area in the diagram below) are completely omitted. To correct this, the term $|A_1 \cap A_2 \cap A_3|$ must be added back:

$$\left|\bigcup_{i=1}^{3} A_i\right| = |A_1 \cup A_2 \cup A_3| =$$
$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|. \quad (3.10)$$



Example 3.35. In a class of 30 students, 20 are learning English, 15 are learning German, and 10 are learning French. Among them, 6 students are learning both English and German, 5 students are learning both English and French, and 6 students are learning both German and French. There are no students who are not learning any language. How many students are learning all three languages?

Let us denote by E the number of students learning English, by G the number of students learning German, and by F the number of students learning French. Then, according to the formula (3.10, p. 64), we have:

$$|E \cup G \cup F| = |E| + |G| + |F| - |E \cap G| - |E \cap F| - |G \cap F| + |E \cap G \cap F|,$$

thus

$$\begin{aligned}|E \cap G \cap F| &= |E \cup G \cup F| - |E| - |G| - |F| + |E \cap G| + |E \cap F| + |G \cap F| = \\ &= 30 - 20 - 15 - 10 + 6 + 5 + 6 = 2,\end{aligned}$$

therefore, 2 students are learning all three languages.

The following theorem, although simple in its statement, is widely used in many areas of mathematics.

Theorem 3.36. (Dirichlet's Box Principle) *If the set $A = A_1 \cup A_2 \cup \cdots \cup A_m$ contains n elements, where $n > m$, then at least one set A_i, $i \in \{1, \ldots, m\}$ contains at least 2 elements.*

The name of the above theorem refers to a less formal but equivalent and more intuitive formulation of this principle:

> If n objects are distributed into m boxes and $n > m$, then at least one box contains at least 2 objects.

With the help of this theorem, one can justify, for example, that among 13 people there must be at least two born in the same month. To do this, it is enough to take 12 boxes labeled with the names of the months and "put" into them the people who were born in each month. Since there are fewer people than boxes, at least one box will contain 2 (or more) people, which means they were born in the same month.

Example 3.37. Prove that among any seven different integers, there are two numbers whose sum or difference is divisible by 10.

Let the boxes be labeled:

$$\{0\}, \{5\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}.$$

We place each of the seven considered numbers in the box labeled with the remainder of that number when divided by 10; note that the numbers appearing in the box labels exhaust all possible remainders when divided by 10. Since there are 6 boxes and 7 numbers, according to Theorem 3.36, at least one of the boxes contains at least two numbers. If it is the box labeled $\{0\}$ or $\{5\}$, then the numbers in that box give a remainder of 0 or 5, respectively, when divided by 10, so both their sum and difference are divisible by 10. The labels of the remaining boxes are of the form $\{k, 10 - k\}$ where $k \in \{1, 2, 3, 4\}$. In the case where one of these boxes contains two numbers, we must consider two scenarios. In the first case, both numbers give a remainder of k when divided by 10, or both give a remainder of $10 - k$. In this case, their difference when divided by 10 will yield a remainder of 0, meaning it will be divisible by 10. In the second case, one of these numbers has the form $10i + k$, while the other has the form $10j + 10 - k$, for some integers $i, j \in \mathbb{Z}$. Thus, their sum takes the form $10i + k + 10j + 10 - k = 10(i + j + 1)$, which is therefore divisible by 10.

3.6. Exercises

Variations without Repetition

Exercise 3.1. Grandma has 5 grandchildren and bought 8 different chocolates in the store. In how many ways can Grandma give sweets to the children if each child is to receive exactly 1 chocolate? (Example 3.7, p. 56.)

Exercise 3.2. In the souvenir shop, we have 9 types of postcards available (each in at least 4 copies). How many ways are there to send 1 postcard to 4 friends, so that each receives a different postcard?

Exercise 3.3. In a certain factory, it was decided to introduce identification badges for the employees working there. Each badge is to consist of 4 different letters written in sequence from the set $\{A, B, C, D, E, F\}$. How many different badges can be created this way?

Exercise 3.4. We are predicting the top 3 places in a competition involving 10 athletes. How many possibilities do we have, considering the order on the podium?

Exercise 3.5. We roll a die 3 times. How many possible outcomes are there in which a different number of spots appears on each roll?

Exercise 3.6. We roll 3 dice: green, red, and blue. How many possible outcomes are there in which each die shows a different number of spots?

Exercise 3.7. In how many ways can we arrange 5 different flowers in 8 different vases such that each vase can hold at most 1 flower?

Exercise 3.8. In how many ways can 4 people be seated on 10 chairs arranged in a single row?

Exercise 3.9. In how many ways can we arrange 7 different tablecloths on 9 numbered tables?

Variations with Repetition

Exercise 3.10. Alicja has a test consisting of 9 questions to solve. For each question, she can select one of 3 answers, or she can leave the question unanswered. In how many ways can Alicja complete the test? (Example 3.9, p. 57.)

Exercise 3.11. How many different 9-digit numbers are there (digits can repeat):

a) composed of digits $\{1, 2, \ldots, 9\}$?

b) composed of digits $\{1, 2, \ldots, 9\}$ that represent the same number regardless of the reading direction?

Exercise 3.12. We flip a coin 8 times, resulting in a sequence of heads and tails. How many possible outcomes are there?

Exercise 3.13. How many different ways are there to paint a chessboard consisting of numbered 8 rows and 8 columns

a) with two colors, b) with three colors,

if we assume that each square must be painted?

Exercise 3.14. How many different ways are there to paint a chessboard consisting of numbered 8 rows and 8 columns

a) with two colors, b) with three colors,

if not all squares must be painted?

Exercise 3.15. In the souvenir shop, we have 9 types of postcards available (each in at least 4 copies). How many ways are there to send 1 postcard to 4 friends?

Exercise 3.16. What is the maximum number of different symbols that can be created in the Braille alphabet? A symbol consists of at least one and no more than six raised dots arranged in three rows and two columns.

Exercise 3.17. A participant in a football betting pool predicts the outcomes of 12 football matches on one ticket. The possible outcomes are: home win, draw, away win. How many tickets must be filled out to ensure at least one correct prediction for all 12 matches?

Exercise 3.18. A cylindrical combination lock has 4 coaxial rings, each containing 10 digits. How many attempts must be made in the worst-case scenario to unlock it?

Exercise 3.19. Each of the 12 convicts is to be placed in one of 7 prisons. In how many ways can the convicts be distributed among these prisons?

Exercise 3.20. To Christopher's treehouse, there are stairs consisting of 8 steps. In how many ways can Christopher paint the steps using the following colors: mint green, yellow, orange, red, and purple? We assume that each step will be painted entirely in one color.

Exercise 3.21. How many different strings composed of 11 beads can be created using any number of red, white, blue, and green beads? Beads of the same color are indistinguishable. We distinguish between the left and right ends of the string.

Exercise 3.22. We roll a die 3 times, recording the number of spots after each roll and resulting in a sequence of three numbers.

a) How many possible outcomes are there?

b) How many outcomes are there where an even number appears on each roll?

Exercise 3.23. We roll 3 dice: green, red, and blue.

a) How many possible outcomes are there?

b) How many outcomes are there where an even number appears on each die?

Exercise 3.24. How many subsets are there for a set with 17 elements? Compare this problem with Example 1.12 (p. 14).

Exercise 3.25. There are 9 grooves carved into a key. Each groove has a depth from 0 mm to 7 mm with a step change in depth of 1 mm. How many different keys can be produced?

Exercise 3.26. In how many ways can we arrange 20 different apples in 5 numbered boxes, assuming that some boxes may remain empty?

Permutations without Repetition

Exercise 3.27. An art exhibition is being prepared in the gallery. On a certain wall, 8 selected paintings by Picasso need to be hung in a row. In how many ways can this be done? (Example 3.11, p. 57.)

Exercise 3.28. How many different 9-digit numbers can be formed using the digits: $1, 2, \ldots, 9$, in which no digit repeats?

Exercise 3.29. Bartek has 4 sisters. In how many ways can he give a tulip, a sunflower, a lily, and a carnation to his sisters if each girl receives exactly 1 flower?

Exercise 3.30. In how many ways can a group of 7 people line up at the ticket counter?

Exercise 3.31. Justyna has 6 windows in her apartment and potted plants: a ficus, a cactus, an orchid, a fern, ivy, and a violet. In how many ways can she arrange the plants if exactly 1 flower must be placed on each windowsill?

Exercise 3.32. On the shelf, there are 5 different books: a detective novel, a romance, a horror story, science fiction, and fantasy. In how many ways can all the books be arranged on the shelf?

Exercise 3.33. Marian has 7 canaries and 7 different cages. In how many ways can he arrange the birds in the cages if there is to be exactly 1 canary in each cage?

Exercise 3.34. We have purchased 4 different postcards. In how many ways can we send 1 postcard to each of 4 friends?

Exercise 3.35. How many different arrangements of the set $A = \{1, 2, \ldots, n\}$ exist, where $n \geq 6$, such that the numbers 5 and 6 are adjacent in their natural order?

Permutations with Repetition

Exercise 3.36. In how many ways can 4 orange juices, 5 apple juices, 3 pineapple juices, and 6 grapefruit juices be arranged in a row on a shelf, assuming that juices of the same flavor are indistinguishable? (Example 3.13, p. 58.)

Exercise 3.37. How many different strings of beads can be created if each string contains 4 red beads, 2 white beads, 3 blue beads, and 2 green beads? Beads of the same color are indistinguishable. We distinguish between the left and right ends of the string.

Exercise 3.38. In how many ways can we arrange on a library shelf 5 books titled "The Deluge," 3 books titled "Sir Michael," and 6 books titled "With Fire and Sword"? Books with the same title are indistinguishable.

Exercise 3.39. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are indistinguishable and that the order of drawing is important. (Example 3.28, p. 62.)

Exercise 3.40. In the game of "Skat" with 32 cards, 10 cards are dealt among 3 players, and the remaining 2 cards are placed in the skat (on the table). How many different ways are there to deal the cards?

Exercise 3.41. How many different 10-digit numbers can we create if we have the digit set: 5, 3, 3, 8, 8, 8, 2, 2, 2, 2?

Exercise 3.42. How many different "words" can be formed using all the letters of the word:

a) ANANAS,

b) MATEMATYKA,

c) KOMBINATORYKA,

d) TOTOLOTEK.

Exercise 3.43. How many different ways are there to paint 10 balls, each of a different size, using the colors: blue, red, and green, such that there are 5 blue balls, 2 red balls, and 3 green balls?

Combinations without Repetition

Exercise 3.44. At a party, there were 14 acquaintances. How many handshakes occurred if each person greeted every other person? (Example 3.15, p. 58.)

Exercise 3.45. On a circle, 15 different points are marked. How many different ninetagon can be drawn with vertices at these points?

Exercise 3.46. The lottery game "Duży Lotek" involves selecting 6 numbers drawn from the numbers 1 to 49. How many tickets must be purchased to ensure a grand prize win?

Exercise 3.47. The commander of a police station has 7 police officers at his disposal. In how many ways can the commander form a 4-person patrol from these officers?

Exercise 3.48. We roll 3 identical dice. How many possible outcomes are there in which each die shows a different number of spots?

Exercise 3.49. In how many ways can 7 identical tablecloths be arranged on 9 numbered tables?

Exercise 3.50. We are predicting the top 3 places in a competition involving 10 athletes. How many possibilities do we have, not considering the order on the podium?

Exercise 3.51. A certain group of students consists of 20 men and 15 women. In how many ways can a subgroup of 8 people be selected?

Exercise 3.52. In the competition, there are 33 participants, among whom are Krzysztof and Marcin. The participants compete in groups of three. In how many ways can Krzysztof choose 2 friends to form a trio?

Exercise 3.53. At a chess tournament, there were 30 participants. How many games were played if each participant played against every other participant?

Exercise 3.54. In how many ways can we arrange 5 identical flowers in 8 different vases such that each vase can hold at most 1 flower?

Combinations with Repetition

Exercise 3.55. How many different sets containing 7 balloons can be formed, having an unlimited number of red, green, and blue balloons? We assume that balloons of the same color are indistinguishable. (Example 3.17, p. 59.)

Exercise 3.56. How many different sets of 11 beads can be created if we have an unlimited number of beads in red, white, blue, and green? Beads of the same color are indistinguishable.

Exercise 3.57. How many different sets of 9 candies can be created if we have an unlimited number of chocolate, mint, fruit, nut, and yogurt candies? Candies of the same flavor are indistinguishable.

Exercise 3.58. In how many ways can we arrange 20 identical apples in 5 numbered boxes, assuming that some boxes may remain empty?

Exercise 3.59. We roll 3 identical dice.

a) How many possible outcomes are there?

b) How many outcomes are there in which each die shows an even number?

Exercise 3.60. In how many ways can we choose 10 balls from an unlimited supply of red, blue, and green balls if we want at least 4 red balls? Balls of the same color are indistinguishable.

Exercise 3.61. We flip 8 identical coins. How many possible outcomes are there?

Exercise 3.62. In how many ways can we choose 5 fruits from 10 identical oranges and 20 identical pears?

The Multiplication Principle

Exercise 3.63. Emil decided to have lunch at the bar consisting of soup, a main course, a salad, and a dessert. In how many ways can Emil compose his meal if he has 2 different soups, 10 different main courses, 7 different salads, and 3 different desserts to choose from? (Example 3.20, p. 60.)

Exercise 3.64. Calculate how many natural divisors the following numbers have (compare with note 2.14 on p. 41):

a) 24, b) 343, c) 25 725, d) 90 000.

Exercise 3.65. Leon has 3 children of different ages, and he reads a book to each of them separately before bed. The youngest child has 5 books, the middle child has 7, and the oldest has 10. In how many ways can Dad choose literature to read to the children tonight?

Exercise 3.66. In the hair salon, there are 5 women: a brunette (brown hair), a black-haired woman, a blonde, a redhead, and an older gray-haired lady. The hairdresser has hair dyes in the following colors: brown, black, blonde, and red. In how many ways can the women's hair be dyed if they want to change their hair color?

Exercise 3.67. How many different 9-digit numbers can be formed using the digits: $0, 1, 2, \ldots, 9$, if digits can repeat?

Exercise 3.68. From the elements of the set $\{0, 1, 2, 3, 4, 5, 6\}$ we create 3-digit numbers. Calculate how many numbers can be formed that are less than 500. We assume that digits can repeat.

Exercise 3.69. We have purchased 4 different colored postcards and 4 different black-and-white postcards. In how many ways can we send each of our 4 friends one colored postcard and one black-and-white postcard?

Exercise 3.70. Justyna has 6 windows in her apartment and plants: a ficus, a cactus, an orchid, a fern, ivy, and a violet. She also has 6 pots: green, red, blue, yellow, purple, and orange. In how many ways can she plant the flowers in the pots and arrange them if exactly 1 plant must be placed on each windowsill?

Exercise 3.71. How many ways are there to arrange 5 men and 4 women in a row such that there are men on both sides of each woman?

Exercise 3.72. Calculate how many ways registration numbers consisting of 2 letters from the set $\{A, B, C, D, E\}$ and 6 arbitrary digits can be formed if:

a) letters and digits can repeat,

b) neither letters nor digits can repeat,

c) letters can repeat but digits cannot repeat,

d) letters cannot repeat but digits can repeat.

Exercise 3.73. In the class there are 27 people: 15 girls and 12 boys. In how many ways can a 3-person representation of the class be selected consisting of 2 girls and 1 boy?

Exercise 3.74. In the class there are 27 students: 15 girls and 12 boys. For St. Nicholas Day, the class organized a lottery with 27 tickets, among which there are exactly 5 winning tickets. In how many ways can we choose 3 girls and 2 boys who won prizes if the prizes are identical? (Example 3.26, p. 61.)

Exercise 3.75. A certain group of students consists of 20 men and 15 women. In how many ways can a subgroup consisting of 6 men and 7 women be selected?

Exercise 3.76. In a box there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be exactly 2 white balls and 4 black balls? We assume that balls of the same color are distinguishable and that the order of drawing is not important. (Example 3.30, p. 63.)

Exercise 3.77. At a certain Czech university in the first year there are 60 students: 8 Poles, 13 Spaniards, and 17 Germans; the remaining people are Czechs. In how many ways can we choose a representative group of five people that includes one person from Poland, Spain, and Germany as well as two people from The Czech Republic?

Exercise 3.78. In lottery game "Duży Lotek," 6 numbers were drawn from numbers ranging from 1 to 49. How many different bets are possible where exactly 3 numbers were correctly predicted?

The Addition Principle

Exercise 3.79. How many committees consisting of 4 people can be formed from a group of 9, if two people from this group, Agnieszka and Bogdan, do not want to be in the same committee? Hint: consider 3 cases. (Example 3.23, p. 60.)

Exercise 3.80. A certain group of students consists of 20 men and 15 women. In how many ways can a subgroup consisting of either 6 men or 7 women be selected? Hint: consider 2 cases.

Exercise 3.81. We roll two dice: a green die and a red die. How many outcomes are there in which the sum of the rolled spots is even? Hint: consider 6 cases.

Exercise 3.82. We roll two dice simultaneously: a black die and a white die. Determine the number of rolls in which:

a) the number of spots on the white die is less than the number of spots on the black die (hint: consider 6 cases),

b) the number of spots on the black die is not less than the number of spots on the white die (hint: consider 6 cases).

Exercise 3.83. In a certain group of 10 people, there are Cezary and Daria. How many ways are there to arrange 10 people in a row such that Cezary and Daria stand next to each other? Hint: consider 2 cases.

Exercise 3.84. In a competition, there are 33 participants, among whom are Krzysztof and Marcin. The participants compete in groups of three. How many groups can be formed that include Krzysztof or Marcin? Hint: consider 3 cases.

Exercise 3.85. In a competition, there are 33 participants, among whom are Krzysztof and Marcin. The participants compete in groups of three. How many groups can be formed that include either Krzysztof or Marcin? Hint: consider 2 cases.

The Multiplication and Addition Principles

Exercise 3.86. A certain group of foreigners consists of 5 Spaniards, 6 Frenchmen, and 8 Italians. In how many ways can a 2-person delegation be selected from this group such that the individuals in the delegation are not of the same nationality? Hint: consider 3 cases.

Exercise 3.87. In how many different ways can we choose a trio of different numbers from the numbers $1, 2, \ldots, 50$, such that their sum is odd? Trios that differ only in order are considered the same. Hint: consider 2 cases.

Exercise 3.88. In lottery game "Duży Lotek," 6 numbers were drawn from the numbers 1 to 49. How many different bets are possible where at least 3 numbers were correctly predicted? Hint: consider 4 cases.

Exercise 3.89. To withdraw money from an ATM, one must use a card and enter a 4-digit PIN on a 10-digit keypad. Filip hasn't used his card in a long time and doesn't remember his PIN exactly, but he recalls that the second and third digits are odd, and the sum of the two outer digits equals 5. Calculate how many times at most Filip will have to enter his PIN to get cash. Remember that 0 is an even digit. Hint: consider 6 cases.

Exercise 3.90. To open a briefcase, one must correctly set 3 small cylinders at the lock. Each cylinder has digits from 0 to 9. President Ireneusz has been on a long vacation and hasn't opened the briefcase in a while, but he remembers that the first and last digits are even, and the middle digit is a prime number. Calculate how many cylinder settings he will have to check in the worst case to open the briefcase. Remember that 0 is an even digit. Hint: consider 4 cases.

Exercise 3.91. Morse code letters are formed from sequences of dots and dashes, where these symbols can repeat. How many letters can be formed using:

a) exactly 4 symbols,

b) at most 4 symbols,

c) no fewer than 3 and no more than 6 symbols.

Exercise 3.92. Each alphanumeric character can be assigned a sequence composed of the digits 0 and 1. If different characters correspond to different sequences, this assignment is called a code. How many characters can be encoded using sequences of no more than eight elements?

Exercise 3.93. From a group of 18 students, consisting of 9 boys and 9 girls, we select a non-empty subset consisting of an equal number of girls and boys. In how many ways can we do this?

Various Problems

Exercise 3.94. From the elements of the set $\{0, 1, 2, 3, 4, 5, 6\}$ we create 3-digit numbers. Calculate how many numbers can be formed that are less than 500 and have 3 different digits.

Exercise 3.95. In a train compartment, there are 8 numbered seats arranged in two opposite rows of 4 seats each. Five people entered the empty compartment: Anna, Beata, Cecylia, Darek, and Edek. The women sat in the same row, while the men sat in the other row. Calculate how many ways these people could take their seats such that:

a) the women are facing the direction of travel?

b) each man has a woman sitting across from him?

Exercise 3.96. In how many ways can we draw 2 cards sequentially without replacement from a deck of 52 cards such that the first card is an ace and the second card is not a queen?

Exercise 3.97. In how many ways can we draw 2 cards sequentially without replacement from a deck of 52 cards such that the first card is a diamond and the second card is not a queen? Hint: consider 2 cases.

Exercise 3.98. How many natural 8-digit numbers exist such that the product of their digits in decimal representation equals 12? Hint: consider 3 cases.

Exercise 3.99. In a certain group of 10 people, there are Cezary and Daria. How many ways can these people be arranged in a line so that Cezary and Daria do not stand next to each other? Hint: consider the opposite situation. (Example 3.32, p. 63.)

Exercise 3.100. We roll three six-sided dice: yellow, red, and green. How many possible situations are there in which the product of the number of spots rolled on the dice is divisible by 3? Hint: how many possible situations exist where the resulting product is not divisible by 3? (Example 3.33, p. 63.)

Exercise 3.101. One winter evening, 9 friends went to their friend Łukasz's house. At the entrance was an empty coat rack where each of them left their coat. When leaving, they randomly put on their coats. How many situations are there in which at least one of them returns home wearing a coat that does not belong to them? Hint: how many possible situations are there where each friend returns home wearing their own coat?

Exercise 3.102. In a competition involving 33 participants, among whom are Krzysztof and Marcin. The participants compete in groups of three. How many groups can be formed that include Krzysztof or Marcin? Hint: how many groups can be formed that do not include either Krzysztof or Marcin?

Exercise 3.103. In how many arrangements of the letters a, b, c, d, e, f, g will the syllable *cad* not appear? Hint: how many arrangements contain the syllable *cad*?

Exercise 3.104. How many diagonals can be drawn in a convex octagon? Hint: how many sides does a convex octagon have?

Exercise 3.105. How many three-digit numbers are there that have at least one digit less than the digits of the number 718? Hint: how many such numbers have all digits greater than or equal to those of the number 718?

Exercise 3.106. How many ways are there to arrange 53 people in a row such that a selected group of 12 people stands next to each other?

Exercise 3.107. On a trip to Disneyland in France, there is a group of 30 children: 5 Chinese, 3 Brazilians, 7 Canadians, and the remaining children are French. In how many ways can all the children be arranged in a row if people from the same country must stand next to each other?

Exercise 3.108. How many ways are there to place 6 different shirts and 5 different sweaters in 4 different drawers?

Exercise 3.109. In a souvenir shop, we have 9 types of postcards (each in at least 4 copies). How many ways are there to send 1 postcard to each of 4 friends such that everyone receives the same postcard?

Exercise 3.110. In how many ways can we distribute 5 identical candies among five children?

Exercise 3.111. In how many ways can we arrange 5 different flowers in 8 identical vases such that each vase can hold at most 1 flower?

Exercise 3.112. In how many ways can we arrange 5 identical flowers in 8 identical vases such that each vase can hold at most 1 flower?

Exercise 3.113. Grandma has 5 grandchildren and bought 8 identical chocolates in the store. In how many ways can Grandma distribute the sweets so that each child receives exactly 1 chocolate?

Exercise 3.114. In how many different ways can 6 different greeting cards be placed into 10 identical envelopes? We can put at most 1 card into each envelope, and some envelopes may remain empty. (Example 3.24, p. 61.)

Exercise 3.115. In how many ways can 9 identical pictures be placed into 8 identical frames? Each frame can contain at most 1 picture, and no picture can be left without a frame. (Example 3.25, p. 61.)

Exercise 3.116. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are indistinguishable and that the order of drawing is not important. (Example 3.29, p. 63.)

Exercise 3.117. We roll 3 identical dice. How many possible outcomes are there in which each die shows the same number of spots?

Exercise 3.118. Bartek has 4 sisters. In how many ways can he give 4 identical roses to his sisters, if each girl is to receive exactly 1 flower?

Exercise 3.119. There are 27 people in the class: 15 girls and 12 boys. In how many ways can a 5-member class representation be chosen, consisting of exactly 2 girls and 1 boy?

Exercise 3.120. A game of bridge starts with dealing 52 cards from a standard deck among 4 players, so that each player receives 13 cards. How many ways are there for a designated player to receive all cards of the same suit?

Exercise 3.121. In how many ways can 7 identical tablecloths be arranged on 9 identical tables?

Exercise 3.122. In how many ways can 7 different tablecloths be arranged on 9 identical tables?

Exercise 3.123. From a deck of 52 cards, we draw x cards without replacement so that there are cards of each suit among them. The order of drawing is not important.

a) In how many ways can this be done if $x = 5$?

b) In how many ways can this be done if $x = 6$?

Exercise 3.124. In a certain town, car license plates start with the letters XY, followed by 5 characters: digits from 0 to 9 or letters from the Latin alphabet (26 letters).

a) How many possible license plates are there where after XY there are only digits, given that there is no plate XY 00000?

b) How many possible license plates are there where after XY there are 2 letters followed by 3 digits?

Exercise 3.125. From a standard deck of 52 playing cards, we draw without replacement 13 cards. How many possible outcomes are there in which we draw exactly 1 ace, exactly 3 kings, and exactly 2 queens? The order in which the cards are drawn is not important.

Exercise 3.126. In how many ways can a non-empty subset of fruits be formed, having at disposal 13 identical bananas and 11 identical plums?

Exercise 3.127. Calculate how many integer divisors the following numbers have (compare with Remark 2.14, p. 41):

a) 24, b) 343, c) 25 725, d) 90 000.

Exercise 3.128. How many 5-digit numbers greater than 20 000 but less than 40 000 can be formed using the digits: 1, 2, 3, 4, 5, if:

a) no digit repeats,

b) the choice of digits is unrestricted?

Exercise 3.129. On a hiking trip, there are 8 girls and 7 boys walking in a line. How many different ways can they be arranged if:

a) they walk alternately: girl, boy, girl, . . . ,

b) the arrangement in line is arbitrary,

c) boys walk together in one group and girls also walk together in one group.

Exercise 3.130. In a classroom, there are 14 numbered two-person desks. In how many ways can 14 girls and 14 boys be seated so that:

a) in each desk, a boy sits on the left side and a girl on the right side,

b) in each desk, there is one boy and one girl.

Exercise 3.131. How many monograms (two-letter initials) can be formed from the letters of the Latin alphabet (26 letters), if:

a) letters in the monogram cannot repeat,

b) letters in the monogram can repeat,

c) letters in the monogram are the same.

Exercise 3.132. In an empty elevator standing on the ground floor, 6 passengers entered. The elevator goes up, stopping at the next 8 floors. We assume that all passengers will exit during this ride. In how many ways can passengers exit the elevator if:

a) no additional conditions are imposed?

b) no two passengers will exit on the same floor?

Exercise 3.133. Cities X and Y are connected by 6 roads, and there are 8 roads between cities Y and Z. In how many ways can one travel

$$X \longrightarrow Y \longrightarrow Z \longrightarrow Y \longrightarrow X,$$

if:

a) no segment of the route can repeat on the return trip,

b) any segment of the route can repeat on the return trip,

c) one must return via the same route.

Exercise 3.134. In a class, there are 27 people: 15 girls and 12 boys. In how many ways can a class trio (chairperson, deputy, treasurer) be chosen so that it consists of 2 girls and 1 boy?

Exercise 3.135. In a class, there are 27 students: 15 girls and 12 boys. For Saint Nicholas Day, the class organized a lottery with 27 tickets, among which there are exactly 5 winning tickets. In how many ways can we choose 3 girls and 2 boys who won prizes if the prizes are distinct? (See Example 3.26, p. 61.)

Exercise 3.136. In how many ways can 6 objects be paired?

Exercise 3.137. In a class, there are 27 people: 15 girls and 12 boys. In how many ways can 3 numbered teams of 9 people be formed if each team must consist of 5 girls and 4 boys?

Exercise 3.138. The class consists of 20 people. The teacher decided to organize work into 5 groups of 4 people each. In how many ways can the division be made if:

a) the groups are numbered from 1 to 5?

b) the groups are not numbered?

(Example 3.27, p. 62.)

Exercise 3.139. In how many ways can 10 people simultaneously hold 5 phone conversations?

Exercise 3.140. In how many ways can 8 people be arranged in 4 numbered rooms for 2 people each?

Exercise 3.141. In how many ways can 8 out of 9 people be arranged in 4 numbered rooms for 2 people each?

Exercise 3.142. In a box, there are 9 white balls and 7 black balls. In how many ways can we draw 6 balls from this box without replacement, among which there will be 2 white balls and 4 black balls? We assume that balls of the same color are distinguishable, and the order of drawing is important. (See Example 3.31, p. 63.)

Exercise 3.143. In how many ways can you choose from 13 couples one woman and one man who

a) are married to each other?

b) are not married to each other?

Exercise 3.144. How many numbers greater than 3 000 000 can be formed using the digits: 1, 2, 2, 4, 6, 6, 6?

Exercise 3.145. We roll a die three times. How many possible outcomes are there in which exactly 2 different numbers of spots appear?

More Difficult Problems

Exercise 3.146. In a class of 25 students, 5 tickets to the theater are being drawn for seats numbered: 15, 16, 17, 18, 19, in the last row. We know that the tickets were won by Waldek and Zenek, but the other 3 winners are still unknown. How many outcomes of the drawing exist where:

a) Waldek and Zenek will sit next to each other,

b) Waldek and Zenek will not sit next to each other.

Exercise 3.147. In how many ways can the spots numbered from 1 to 6 be arranged on a cubic die if:

a) all faces of the die are painted in different colors?

b) all faces of the die are painted the same color?

Exercise 3.148. In commonly used cubic dice, the spots are arranged such that the sum of the spots on opposite faces equals 7. How many ways can the spots numbered from 1 to 6 be arranged on such a die if:

a) all faces of the die are painted in different colors?

b) all faces of the die are painted the same color?

Exercise 3.149. In how many ways can we seat 6 men and 6 women at a round table with 12 seats such that no two people of the same gender sit next to each other?

Exercise 3.150. In how many ways can we place 2 kings (one white and one black) on an $m \times n$ chessboard, where $m, n \geq 3$, so that they do not stand on adjacent squares? Adjacent squares are defined as those sharing a side or a corner. Hint: consider 3 cases.

Exercise 3.151. Let $A = \{1, 2, \ldots, k\}$ and $B = \{1, 2, \ldots, n\}$ where $k \leq n$. Let $f : A \longrightarrow B$ be a function. How many different:

a) functions f exist?

b) strictly increasing functions f exist?

c) non-decreasing functions f exist?

d) injective functions f exist?

e) constant functions f exist?

3.7. Answers

Answer 3.1. $\dfrac{8!}{(8-5)!} = \dfrac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

Answer 3.2. $\dfrac{9!}{(9-4)!} = \dfrac{9!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6$

Answer 3.3. $\dfrac{6!}{(6-4)!} = \dfrac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3$

Answer 3.4. $\dfrac{10!}{(10-3)!} = \dfrac{10!}{7!} = 10 \cdot 9 \cdot 8$

Answer 3.5. $\dfrac{6!}{(6-3)!} = \dfrac{6!}{3!} = 6 \cdot 5 \cdot 4$

Answer 3.6. $\dfrac{6!}{(6-3)!} = \dfrac{6!}{3!} = 6 \cdot 5 \cdot 4$

Answer 3.7. $\dfrac{8!}{(8-5)!} = \dfrac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

Answer 3.8. $\dfrac{10!}{(10-4)!} = \dfrac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$

Answer 3.9. $\dfrac{9!}{(9-7)!} = \dfrac{9!}{2!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

Answer 3.10. 4^9

Answer 3.11.

a) 9^9,

b) 9^5.

Answer 3.12. 2^8

Answer 3.13.

a) 2^{64}, b) 3^{64}.

Answer 3.14.

a) 3^{64}, b) 4^{64}.

Answer 3.15. 9^4

Answer 3.16. $2^6 - 1$

Answer 3.17. 3^{12}

Answer 3.18. 10^4

Answer 3.19. 7^{12}

Answer 3.20. 5^8

Answer 3.21. 4^{11}

Answer 3.22.

a) 6^3,

b) 3^3.

Answer 3.23.

a) 6^3,

b) 3^3.

Answer 3.24. 2^{17}

Answer 3.25. 8^9

Answer 3.26. 5^{20}

Answer 3.27. $8!$

Answer 3.28. $9!$

Answer 3.29. $4!$

Answer 3.30. $7!$

Answer 3.31. $6!$

Answer 3.32. $5!$

Answer 3.33. $7!$

Answer 3.34. $4!$

Answer 3.35. $(n-1)!$

Answer 3.36. $\dfrac{18!}{4!\cdot 5!\cdot 3!\cdot 6!}$

Answer 3.37. $\dfrac{11!}{4!\cdot 2!\cdot 3!\cdot 2!}$

Answer 3.38. $\dfrac{14!}{5!\cdot 3!\cdot 6!}$

Answer 3.39. $\dfrac{6!}{2!\cdot 4!}$

Answer 3.40. $\dfrac{32!}{10!\cdot 10!\cdot 10!\cdot 2!}$

Answer 3.41. $\dfrac{10!}{1!\cdot 2!\cdot 3!\cdot 4!}$

Answer 3.42.

a) $\dfrac{6!}{3!\cdot 2!\cdot 1!}$,

b) $\dfrac{10!}{2!\cdot 3!\cdot 2!\cdot 1!\cdot 1!\cdot 1!}$,

c) $\dfrac{13!}{2!\cdot 2!\cdot 1!\cdot 1!\cdot 1!\cdot 1!\cdot 2!\cdot 1!\cdot 1!\cdot 1!}$,

d) $\dfrac{9!}{3!\cdot 3!\cdot 1!\cdot 1!\cdot 1!}$.

Answer 3.43. $\dfrac{10!}{5!\cdot 2!\cdot 3!}$

Answer 3.44. $\dbinom{14}{2}$

Answer 3.45. $\dbinom{15}{9}$

Answer 3.46. $\dbinom{49}{6}$

Answer 3.47. $\dbinom{7}{4}$

Answer 3.48. $\dbinom{6}{3}$

Answer 3.49. $\dbinom{9}{7}$

Answer 3.50. $\binom{10}{3}$

Answer 3.51. $\binom{20+15}{8} = \binom{35}{8}$

Answer 3.52. $\binom{33-1}{2}$

Answer 3.53. $\binom{30}{2}$

Answer 3.54. $\binom{8}{5}$

Answer 3.55. $\binom{3+7-1}{7} = \binom{9}{7}$

Answer 3.56. $\binom{4+11-1}{11} = \binom{14}{11}$

Answer 3.57. $\binom{5+9-1}{9} = \binom{13}{9}$

Answer 3.58. $\binom{5+20-1}{20} = \binom{24}{20}$

Answer 3.59.

a) $\binom{6+3-1}{3} = \binom{8}{3}$,

b) $\binom{3+3-1}{3} = \binom{5}{3}$.

Answer 3.60. $\binom{3+6-1}{6} = \binom{8}{6}$

Answer 3.61. $\binom{2+8-1}{8} = \binom{9}{8}$

Answer 3.62. $\binom{2+5-1}{5} = \binom{6}{5} = 6$

Answer 3.63. $\binom{2}{1} \cdot \binom{10}{1} \cdot \binom{7}{1} \cdot \binom{3}{1} = 2 \cdot 10 \cdot 7 \cdot 3$

Answer 3.64.

a) $\binom{4}{1} \cdot \binom{2}{1} = 4 \cdot 2$,

b) $\binom{4}{1} = 4$,

c) $\binom{2}{1} \cdot \binom{3}{1} \cdot \binom{4}{1} = 2 \cdot 3 \cdot 4$,

d) $\binom{5}{1} \cdot \binom{3}{1} \cdot \binom{4}{1} = 5 \cdot 3 \cdot 5$.

Answer 3.65. $\binom{5}{1} \cdot \binom{7}{1} \cdot \binom{10}{1} = 5 \cdot 7 \cdot 10$

Answer 3.66. $\binom{3}{1}^4 \cdot \binom{4}{1} = 3^4 \cdot 4$

Answer 3.67. $\binom{9}{1} \cdot \binom{10}{1}^8 = 9 \cdot 10^8$

Answer 3.68. $\binom{4}{1} \cdot \binom{7}{1} \cdot \binom{7}{1} = 4 \cdot 7 \cdot 7$

Answer 3.69. $4! \cdot 4!$

Answer 3.70. $6! \cdot 6!$

Answer 3.71. $5! \cdot 4!$

Answer 3.72.

a) $5^2 \cdot 10^6$,

b) $\dfrac{5!}{(5-2)!} \cdot \dfrac{10!}{(10-6)!} = \dfrac{5!}{3!} \cdot \dfrac{10!}{4!} = 5 \cdot 4 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$,

c) $5^2 \cdot \dfrac{10!}{(10-6)!} = 5^2 \cdot \dfrac{10!}{4!} = 5^2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$,

d) $\dfrac{5!}{(5-2)!} \cdot 10^6 = \dfrac{5!}{3!} \cdot 10^6 = 5 \cdot 4 \cdot 10^6$.

Answer 3.73. $\binom{15}{2} \cdot \binom{12}{1} = \binom{15}{2} \cdot 12$

Answer 3.74. $\binom{15}{3} \cdot \binom{12}{2}$

Answer 3.75. $\binom{20}{6} \cdot \binom{15}{7}$

Answer 3.76. $\binom{9}{2} \cdot \binom{7}{4}$

Answer 3.77. $\binom{8}{1} \cdot \binom{13}{1} \cdot \binom{17}{1} \cdot \binom{60-8-13-17}{2} = 8 \cdot 13 \cdot 17 \cdot \binom{22}{2}$

Answer 3.78. $\binom{6}{3} \cdot \binom{49-6}{3} = \binom{6}{3} \cdot \binom{43}{3}$

Answer 3.79. $\binom{7}{3} + \binom{7}{3} + \binom{7}{4}$

Answer 3.80. $\binom{20}{6} + \binom{15}{7}$

Answer 3.81. $3+3+3+3+3+3 = 6 \cdot 3$

Answer 3.82.

a) $0+1+2+3+4+5$,

b) $1+2+3+4+5+6$.

Answer 3.83. $9!+9! = 2\cdot 9!$

Answer 3.84. $\binom{33-2}{2}+\binom{33-2}{2}+\binom{33-2}{1} = 2\cdot\binom{31}{2}+31$

Answer 3.85. $\binom{33-2}{2}+\binom{33-2}{2} = 2\cdot\binom{31}{2}$

Answer 3.86. $\binom{5}{1}\cdot\binom{6}{1}+\binom{5}{1}\cdot\binom{8}{1}+\binom{6}{1}\cdot\binom{8}{1} = 5\cdot 6+5\cdot 8+6\cdot 8$

Answer 3.87. $\binom{25}{3}+\binom{25}{1}\cdot\binom{25}{2} = \binom{25}{3}+25\cdot\binom{25}{2}$

Answer 3.88.

$$\binom{6}{3}\cdot\binom{49-6}{3}+\binom{6}{4}\cdot\binom{49-6}{2}+\binom{6}{5}\cdot\binom{49-6}{1}+\binom{6}{6}\cdot\binom{49-6}{0} =$$
$$\binom{6}{3}\cdot\binom{43}{3}+\binom{6}{4}\cdot\binom{43}{2}+\binom{6}{5}\cdot\binom{43}{1}+\binom{6}{6}\cdot\binom{43}{0}$$

Answer 3.89. $5\cdot 5+5\cdot 5+5\cdot 5+5\cdot 5+5\cdot 5+5\cdot 5 = 6\cdot 5\cdot 5$

Answer 3.90. $5\cdot 5+5\cdot 5+5\cdot 5+5\cdot 5 = 4\cdot 5\cdot 5$

Answer 3.91.

a) 2^4,

b) $2+2^2+2^3+2^4$,

c) $2^3+2^4+2^5+2^6$.

Answer 3.92. $\displaystyle\sum_{i=1}^{8} 2^i$

Answer 3.93. $\displaystyle\sum_{k=1}^{9} \binom{9}{k}^2$

Answer 3.94. $\binom{4}{1}\cdot\binom{7-1}{1}\cdot\binom{7-2}{1} = 4\cdot 6\cdot 5$

Answer 3.95.

a) $\dfrac{4!}{(4-3)!}\cdot\dfrac{4!}{(4-2)!} = (4\cdot 3\cdot 2)\cdot(4\cdot 3) =$

b) $\dfrac{4!}{(4-3)!}\cdot\dfrac{3!}{(3-2)!}+\dfrac{4!}{(4-3)!}\cdot\dfrac{3!}{(3-2)!} = (4\cdot 3\cdot 2)\cdot(3\cdot 2)+(4\cdot 3\cdot 2)\cdot(3\cdot 2)$
$= 2\cdot(4\cdot 3\cdot 2)\cdot(3\cdot 2)$

Answer 3.96. $\binom{4}{1} \cdot \binom{52-1-4}{1} = 4 \cdot 47$

Answer 3.97. $1 \cdot \binom{52-4}{1} + \binom{13-1}{1} \cdot \binom{52-4-1}{1} = 48 + 12 \cdot 47$

Answer 3.98. $\binom{8}{1} \cdot \binom{7}{1} + \binom{8}{1} \cdot \binom{7}{1} + \binom{8}{2} \cdot \binom{6}{1} = 8 \cdot 7 + 8 \cdot 7 + \binom{8}{2} \cdot 6$

Answer 3.99. $10! - 2 \cdot 9!$

Answer 3.100. $6^3 - 4^3$

Answer 3.101. $9! - 1$

Answer 3.102. $\binom{33}{3} - \binom{33-2}{3} = \binom{33}{3} - \binom{31}{3}$

Answer 3.103. $7! - 5 \cdot 4! = 7! - 5!$

Answer 3.104. $\binom{8}{2} - 8$

Answer 3.105. $900 - 3 \cdot 9 \cdot 2$

Answer 3.106. $12! \cdot 42!$

Answer 3.107. $5! \cdot 3! \cdot 7! \cdot (30-5-3-7)! \cdot 4! = 5! \cdot 3! \cdot 7! \cdot 15! \cdot 4!$

Answer 3.108. $4^6 \cdot 4^5$

Answer 3.109. 9

Answer 3.110. 1

Answer 3.111. 1

Answer 3.112. 1

Answer 3.113. 1

Answer 3.114. 1

Answer 3.115. 0

Answer 3.116. 1

Answer 3.117. 6

Answer 3.118. 1

Answer 3.119. 0

Answer 3.120. 4

Answer 3.121. 1

Answer 3.122. 1

Answer 3.123.

a) $\binom{4}{1} \cdot \binom{13}{2} \cdot \binom{13}{1} \cdot \binom{13}{1} \cdot \binom{13}{1}$,

b) $\binom{4}{2} \cdot \binom{13}{2} \cdot \binom{13}{2} \cdot \binom{13}{1} \cdot \binom{13}{1} + \binom{4}{1} \cdot \binom{13}{3} \cdot \binom{13}{1} \cdot \binom{13}{1} \cdot \binom{13}{1}$.

Answer 3.124.

a) $10^5 - 1$,

b) $26^2 \cdot 10^3$.

Answer 3.125. $\binom{4}{1} \cdot \binom{4}{3} \cdot \binom{4}{2} \cdot \binom{52-12}{7}$

Answer 3.126. $14 \cdot 12 - 1$

Answer 3.127.

a) $2 \cdot \binom{4}{1} \cdot \binom{2}{1} = 2 \cdot 4 \cdot 2$,

b) $2 \cdot \binom{4}{1} = 2 \cdot 4$,

c) $2 \cdot \binom{2}{1} \cdot \binom{3}{1} \cdot \binom{4}{1} = 2 \cdot 2 \cdot 3 \cdot 4$,

d) $2 \cdot \binom{5}{1} \cdot \binom{3}{1} \cdot \binom{4}{1} = 2 \cdot 5 \cdot 3 \cdot 5$.

Answer 3.128.

a) $2 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2 \cdot 4!$,

b) $2 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 2 \cdot 5^4$.

Answer 3.129.

a) $8! \cdot 7!$,

b) $15!$,

c) $2 \cdot 8! \cdot 7!$.

Answer 3.130.

a) $14! \cdot 14!$,

b) $14! \cdot 14! \cdot 2^{14}$.

Answer 3.131.

a) $26 \cdot 25$,

b) 26^2,

c) 26.

Answer 3.132.

a) 8^6,

b) $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

Answer 3.133.

a) $6 \cdot 8 \cdot 5 \cdot 7$,

b) $6 \cdot 8 \cdot 6 \cdot 8$,

c) $6 \cdot 8$.

Answer 3.134. $\binom{15}{2} \cdot \binom{12}{1} \cdot 3! = \binom{15}{2} \cdot 12 \cdot 3!$

Answer 3.135. $\binom{15}{3} \cdot \binom{12}{2} \cdot 5!$

Answer 3.136. $\dfrac{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{3!} = 5 \cdot 3 \cdot 1$

Answer 3.137. $\binom{15}{5} \cdot \binom{12}{4} \cdot \binom{10}{5} \cdot \binom{8}{4} \cdot \binom{5}{5} \cdot \binom{4}{4}$

Answer 3.138.

a) $\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}$,

b) $\dfrac{\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}}{5!} = \binom{19}{3} \cdot \binom{15}{3} \cdot \binom{11}{3} \cdot \binom{7}{3} \cdot \binom{3}{3}$.

Answer 3.139. $\dfrac{\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{5!} = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$

Answer 3.140. $\binom{8}{2} \cdot \binom{8-2}{2} \cdot \binom{8-4}{2} \cdot \binom{8-6}{2} = \binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}$

Answer 3.141. $\binom{9}{2} \cdot \binom{9-2}{2} \cdot \binom{9-4}{2} \cdot \binom{9-6}{2} = \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot \binom{3}{2}$

Answer 3.142. $\binom{9}{2} \cdot \binom{7}{4} \cdot 6!$

Answer 3.143.

a) 13,

b) $13 \cdot 12$.

Answer 3.144. $\dfrac{6!}{1! \cdot 2! \cdot 3!} + \dfrac{6!}{1! \cdot 2! \cdot 2! \cdot 1!}$

Answer 3.145. $\binom{3}{2} \cdot 6 \cdot 5$.

Answer 3.146.

a) $\binom{23}{3} \cdot 4! \cdot 2,$

b) $\binom{23}{3} \cdot 5! - \binom{23}{3} \cdot 4! \cdot 2.$

Answer 3.147.

a) $6!,$

b) $(1+4) \cdot 3! = 5 \cdot 3!.$

Answer 3.148.

a) $6 \cdot 4 \cdot 2 = 2^3 \cdot 3!,$

b) $2.$

Answer 3.149. $\dfrac{2 \cdot 6! \cdot 6!}{2 \cdot 6} = 5! \cdot 6!$

Answer 3.150. $4(mn-4) + [2(m-2) + 2(n-2)](mn-6) + (m-2)(n-2)(mn-9)$

Answer 3.151.

a) $n^k,$

b) $\binom{n}{k},$

c) $\binom{n+k-1}{k},$

d) $\dfrac{n!}{(n-k)!},$

e) $n.$

Chapter 4

Sample Exam Questions

Question 4.1. Assume that the statement $p \Rightarrow q$ is false. Provide the truth values of the statements:

a) $p \wedge q$,　　b) $p \vee q$,　　c) $q \Rightarrow p$.

Question 4.2. For which values of $p_1, p_2, \ldots, p_n$ does the statement:

a) $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ have a value of 1,

b) $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ have a value of 0,

c) $p_1 \vee p_2 \vee \cdots \vee p_n$ have a value of 1,

d) $p_1 \vee p_2 \vee \cdots \vee p_n$ have a value of 0.

Question 4.3. Derive the law of negation of equivalence.

Question 4.4. Let $A, B \subseteq \Omega$. Determine the sets:

a) $A \cup (A \cap B)$,　　b) $A \cap (A \cup B)$,　　c) $A \cup A'$,　　d) $A \cap A'$.

Question 4.5. Provide an example of non-empty sets A and B such that $A \times B = B \times A$.

Question 4.6. Give an example of a relation that:

a) is symmetric but not antisymmetric,

b) is not symmetric but is antisymmetric,

c) is neither symmetric nor antisymmetric,

d) is symmetric and antisymmetric.

Question 4.7. Let R be a symmetric relation, but not reflexive. Check if the relation R can be a transitive relation.

Question 4.8. Determine the number of factors in the expression $\prod\limits_{\substack{i,j \geq 0 \\ i+j=10}} a_{i,j}$.

Question 4.9. Expand the following expressions:

a) $\sum_{k=1}^{3}\prod_{j=1}^{4} a_{k,j}$,

c) $\sum_{k=1}^{3}\prod_{j=1}^{4}(a_k + b_j)$,

e) $\sum_{k=1}^{3}\prod_{j=1}^{4}(a_k \cdot b_j)$,

b) $\prod_{k=1}^{3}\sum_{j=1}^{4} a_{k,j}$,

d) $\prod_{k=1}^{3}\sum_{j=1}^{4}(a_k + b_j)$,

f) $\prod_{k=1}^{3}\sum_{j=1}^{4}(a_k \cdot b_j)$.

Question 4.10. Draw the graphs of the functions $f_1(x) = \lfloor x \rfloor$, $f_2(x) = \lceil x \rceil$, $f_3(x) = \langle x \rangle$.

Question 4.11. Answer whether there exists a number $x \in \mathbb{R}$ such that $\lceil x \rceil - \lfloor x \rfloor = 2$.

Question 4.12. Answer the following questions.

a) Is the sum/difference/product of two even numbers an even number?

b) Is the sum/difference/product of two odd numbers an odd number?

c) Is the sum/difference/product of two numbers divisible by $k \in \mathbb{N}$ also divisible by k?

d) Is the sum/difference/product of two numbers that give a remainder r when divided by $k \in \mathbb{N}$ also a number that gives a remainder r when divided by k?

Question 4.13. Answer the following questions. Consider all possibilities.

a) Is the sum/product of 3 even numbers an even number?

b) Is the sum/product of 3 odd numbers an odd number?

c) Is the sum/product of 3 numbers divisible by $k \in \mathbb{N}$ also divisible by k?

d) Is the sum/product of 3 numbers that give a remainder r when divided by $k \in \mathbb{N}$ also a number that gives a remainder r when divided by k?

Question 4.14. Formulate the divisibility rules for the numbers: 28, 39, and 42.

Question 4.15. Describe the workings of the Sieve of Eratosthenes.

Question 4.16. Answer the following questions. Consider all possibilities.

a) Is the sum/difference/product of two prime numbers a prime number?

b) Is the sum/difference/product of two composite numbers a composite number?

Question 4.17. Answer the following questions. Consider all possibilities.

a) Is the sum/product of 3 prime numbers a prime number?

b) Is the sum/product of 3 composite numbers a composite number?

Question 4.18. Answer the questions:

a) Does every natural number $n > 1$ have a prime divisor?

b) Does every natural number $n > 1$ have a proper prime divisor?

Question 4.19. For fixed numbers $a_1, a_2, \ldots, a_k$:

a) What is the smallest possible value of $GCD(a_1, a_2, \ldots, a_k), k \in \mathbb{N}$?

b) What is the largest possible value of $GCD(a_1, a_2, \ldots, a_k), k \in \mathbb{N}$?

c) What is the largest possible value of $LCM(a_1, a_2, \ldots, a_k), k \in \mathbb{N}$?

d) What is the smallest possible value of $LCM(a_1, a_2, \ldots, a_k), k \in \mathbb{N}$?

Question 4.20. Consider the formula:

$$GCD(a_1, a_2, a_3) \cdot LCM(a_1, a_2, a_3) = |a_1 \cdot a_2 \cdot a_3|.$$

a) Is the above formula true for any numbers $a_1, a_2, a_3 \in \mathbb{N}$?

b) Are there numbers $a_1, a_2, a_3 \in \mathbb{N}$ for which the above formula is true?

Question 4.21. Justify why the Euclidean algorithm will terminate after a finite number of steps.

Question 4.22. Answer the questions:

a) If the numbers $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ are relatively prime, are they pairwise relatively prime?

b) If the numbers $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ are pairwise relatively prime, are they relatively prime?

Question 4.23. Provide the value of Euler's totient function for a prime number $p \in \mathbb{P}$.

Question 4.24. Provide the value of Euler's totient function for the product $p \cdot q$, where $p, q \in \mathbb{P}$.

Question 4.25. Answer the questions:

a) What element is located in row 15 at position 9 in Pascal's triangle?

b) What element is located in row 23 at position 39 in Pascal's triangle?

c) What is the sum of the elements in rows 4, 5, and 6 of Pascal's triangle?

d) What relationship do the obtained sums have with the numbers 4, 5, and 6?

e) Do the remaining rows also have this property? Justify your answer.

Question 4.26. Expand the expression $(a \pm b)^6$.

Question 4.27. Answer the questions:

a) Consider an n-element permutation with repetitions of the set $A = \{a_1, a_2, \ldots, a_k\}$, where the number of repetitions of each element $a_i \in A$, $i \in \{1, 2, \ldots, k\}$, is equal to 1. To what does this permutation reduce?

b) Consider a k-element combination without repetitions from an n-element set, where $k < n$, whose elements are arranged in a sequence. To what does this combination reduce?

c) Consider a k-element variation without repetitions from an n-element set, where $k = n$. To what does this variation reduce?

d) From a combinatorial perspective, how can we name an element of the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$, where $A_1 = A_2 = \cdots = A_n$ are finite sets?

e) Consider the principle of inclusion-exclusion for a collection of finite pairwise disjoint sets. To what does this principle reduce in this situation?

Question 4.28. Calculate how many people are in a certain group where everyone sings, paints, or programs. There are 50 singers, 45 painters, and 40 programmers. Additionally, 27 people both sing and paint, 14 people both sing and program, and 24 people both paint and program. It is also known that the number of people with all three skills is eight times less than the total number of people in the group.

Question 4.29. Write the principle of inclusion-exclusion for 4 sets.

Question 4.30. Justify that in a group of 8 people, there are at least 2 people born on the same day of the week.

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