Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka V, Częstochowa 1997

Some Applications of the Resolution on Hypergraphs

Adam Kolany

Abstract

We show here some applications of the hypergraph resolution. The presented methods origin from papers of Cowen [1] and Kolany [4].

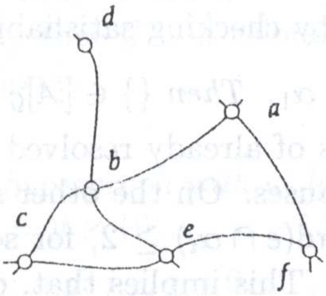
Hypergraph satisfiability and a generalized resolution rule.

A hypergraph is a structure $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is any set and $\mathcal{E}$ a family of nonempty subsets of $\mathcal{V}$. In the sake of simplicity, we shall assume that $\mathcal{V}$ (and hence also $\mathcal{E}$) is finite. Hypergraphs whose edges are 2-element are called graphs. The elements of $\mathcal{V}$ will be called vertices of the hypergraph $\mathcal{G}$ and the elements of $\mathcal{E}$ — its edges. Sets of vertices which do not contain edges will be called $\mathcal{G}$-consistent, or simply consistent, if there is no possibility of misunderstanding. Sets which are not consistent are inconsistent. Sets of vertices will sometimes be called clauses.

Let $\mathcal{A}$ be a family of clauses and let σ be a consistent set of vertices. We will say that σ satisfies $\mathcal{A}$ with respect to $\mathcal{G}$ iff $\sigma \cap \alpha \neq \emptyset$, for every $\alpha \in \mathcal{A}$, (see [1,4]). A family of clauses is satisfiable iff some consistent σ satisfies it. We easily notice that colorability of a graph is equivalent to satisfiability of the family of its all edges.

EXAMPLE. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, *where*

$$\mathcal{V} = \{a, b, c, d, e, f\} \text{ and } \mathcal{E} = \{\{a, f\}, \{a, b, c\}, \{c, e, f\}, \{b, d, e\}\}.$$



Then, the family of clauses $\mathcal{A}_0 = \{\{b\}, \{d\}, \{a, c\}, \{c, f\}, \{e, f\}\}$ *is satisfied by* $\sigma_0 = \{b, d, c, f\}$, *but* $\mathcal{A}_1 = \mathcal{A}_0 \cup \{\{a, e\}\}$ *is not satisfiable. Let us, oppositely, suppose that some* σ *satisfies* $\mathcal{A}_1$. *Then* $b, d \in \sigma$. *Hence* $e \notin \sigma$. *Since* $\{a, e\} \in \mathcal{A}_1$, *we have* $a \in \sigma$ *and since* $\{a, f\} \in \mathcal{E}$, *we get* $f \notin \sigma$. *Then neither of* e, f *is in* σ, *though* $\{e, f\} \in \mathcal{A}_1$. *Contradiction.* $\mathcal{A}_1$ *is not satisfiable.*

The following duality property of hypergraph satisfiability has been noticed by Cowen in [2]:

THEOREM. (Duality Principle) *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and let* $\mathcal{A}$ *be a family of nonempty clauses. Then* $\mathcal{A}$ *is satisfiable wrt.* $\mathcal{G}$ *iff* $\mathcal{E}$ *is satisfiable wrt.* $(\mathcal{V}, \mathcal{A})$.
The following notions can be found in [1,5].

Let $e = \{a_1, \ldots, a_n\}$ be an edge and let $\alpha_1, \ldots, \alpha_n$ be clauses. Then we say that the clause $\alpha = \bigcup_{j=1}^{n}(\alpha_j \setminus \{a_j\})$ results by the resolution on the edge e from the clauses $\alpha_1, \ldots, \alpha_n$. We write then $\alpha_1, \ldots, \alpha_n \vdash_e \alpha$.
If $\mathcal{A}$ is a family of clauses, then the least $\mathcal{A}_0$ closed on the resolution rule and containing $\mathcal{A}$ will be denototed as $[\mathcal{A}]_{\mathcal{G}}$. Since the latter set depends merely on the family $\mathcal{A}$ and the set of edges in fact, we shall also denote it as $[\mathcal{A}]_{\mathcal{E}}$.

The following has been proved in [4]:

THEOREM. *Let* $\mathcal{A}$ *be a family of clauses. Then* $\mathcal{A}$ *is satisfiable iff*

$$\{\} \notin [\mathcal{A}]_{\mathcal{G}}.$$

EXAMPLE. *Let $\mathcal{G}$ and $\mathcal{A}_1$ be as in the first example. We have*

$$\{a,e\},\{e,f\} \vdash_{\{a,f\}} \{e\} \quad \text{and} \quad \{b\},\{d\},\{e\} \vdash_{\{b,d,e\}} \{\}.$$

Hence $\{\} \in [\mathcal{A}_1]_{\mathcal{G}}$, which proves unsatisfiability of $\mathcal{A}_1$.
The following can be helpful by checking satisfiability by the resolution:

REMARK. *If $\alpha \in \mathcal{A}$ and $\alpha \subseteq \alpha_1$. Then $\{\} \in [\mathcal{A}]_{\mathcal{G}}$ iff $\{\} \in [\mathcal{A} \cup \{\alpha_1\}]_{\mathcal{G}}$.*

This lets us omit oversets of already resolved clauses, while searching satisfiability of a family of clauses. On the other side, if $e = \{a_1, \dots, a_n\}$, $a_j \in \alpha_j$, $j = 1, \dots, n$ and $\mathsf{Card}(e \cap \alpha_i) \geq 2$, for some $i = 1, \dots, n$, and if $\alpha_1, \dots, \alpha_n \vdash_e \alpha$, then $\alpha_i \subseteq \alpha$. This implies that, checking satisfiability by resolution, we can restrict ourselves to clauses with one-element meets with the edge we resolve on.

Applications.

In this section we show some applications of hypergraph resolution in deciding the existence of certain objects. Proofs of most of the facts cited below can be found in [4].

1. (Hyper)graph 2-colorability

let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. We say that $\mathcal{G}$ is 2-colorable (or simply colorable), if there exists a function $\kappa : \mathcal{V} \to \{0,1\}$ with the property that $\kappa \| e$ has at least two different elements, for every non-singleton edge e of $\mathcal{G}$. We have:

THEOREM. *Let $\mathcal{G}$ be a hypergraph with no singleton edges. Then $\mathcal{G}$ is colorable iff $\mathcal{E}$ is satisfiable with respect to $\mathcal{G}$.*

EXAMPLE. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be such that*

$$\mathcal{V} = \{a,b,c,d,e\} \text{ and } \mathcal{E} = \{\{a,b,c\},\{a,d,e\},\{b,d\},\{c,e\}\}.$$

In order to decide its colorability, we must check whether $[\mathcal{E}]_{\mathcal{G}}$ contains the empty clause. Because $\{a,b,c\},\{b,d\},\{c,e\} \vdash_{\{a,d,e\}} \{b,c\}$ and $\{a,d,e\}$, $\{c,e\},\{b,d\} \vdash_{\{a,c,b\}} \{d,e\}$, by the Remark at the end of the first section, $\{\} \in [\mathcal{E}]_{\mathcal{G}}$ iff $\{\} \in [\mathcal{A}]_{\mathcal{G}}$, where $\mathcal{A} = \{\{b,d\},\{b,c\},\{c,e\},\{d,e\}\}$. It is however easy to see that resolving from $\mathcal{A}$ yields supersets of clauses of $\mathcal{A}$ only. Hence $\mathcal{E}$ is satisfiable and thus $\mathcal{G}$ is colorable.

Some similar method of deciding the 2-colorability was also considered in [5].

2. n-colorability

A generalisation of colorability is n-colorability of hypergraphs. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. A function $\kappa : \mathcal{V} \to \{0, \ldots, n-1\}$ is an n-coloring of $\mathcal{G}$ iff $Card\ \kappa\|e \geq 2$, $e \in \mathcal{E}$, unless e is a singleton. We say that $\mathcal{G}$ is n-colorable iff there exists an n-coloring of $\mathcal{G}$. We have:

THEOREM. *Let $\mathcal{G}$ be a hypergraph with at least 2-element edges and let $\mathcal{G}^\circ = (\mathcal{V}^\circ, \mathcal{E}^\circ)$, where $\mathcal{V}^\circ = \mathcal{V} \times \{0, \ldots, n-1\}$ and*

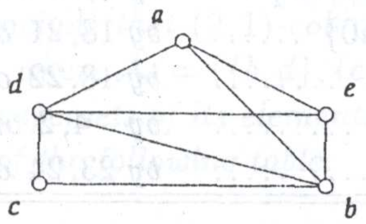
$$\mathcal{E}^\circ = \{\{(v,i),(v,j)\} : i \neq j, i, j = 0, \ldots, n-1\} \cup \bigcup \{e \times \{j\} : e \in \mathcal{E}, j = 0, \ldots, n-1\}.$$

Then $\mathcal{G}$ is n-colorable iff the family $\mathcal{A}^\circ = \{\{v\} \times \{0, \ldots, n-1\} : v \in \mathcal{V}\}$ is satisfiable with respect to $\mathcal{G}^\circ$.

EXAMPLE. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{a, b, c, d, e\}$ and*

$$\mathcal{E} = \{\{a,b\}, \{a,e\}, \{a,d\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{d,e\}\}$$

(hence $\mathcal{G}$ is a graph in fact).



We will decide whether $\mathcal{G}$ is 3-colorable. Instead of (v,i), we will write vi, $v \in \mathcal{V}$, $i = 0, 1, 2$, in the following. We have:

1.	$\{a0, a1, a2\}$,	
2.	$\{b0, b1, b2\}$,	
3.	$\{c0, c1, c2\}$,	
4.	$\{d0, d1, d2\}$,	
5.	$\{e0, e1, e2\}$,	
6.	$\{b0, b2, c0, c2\}$	*by* 2, 3 *on* $\{b1, c1\}$,
7.	$\{d1, d2, c1, c2\}$	*by* 3, 4 *on* $\{d0, c0\}$,
8.	$\{d1, d2, b0, b2, c2\}$	*by* 6, 7 *on* $\{c0, c1\}$,
9.	$\{d1, d2, e1, e2\}$	*by* 4, 5 *on* $\{d0, e0\}$,
10.	$\{c0, c1, e0, e1\}$	*by* 3, 5 *on* $\{c2, e2\}$,
11.	$\{c0, c1, d1, d2, e1\}$	*by* 9, 10 *on* $\{e0, e2\}$,
12.	$\{c1, d1, d2, b0, b2, e1\}$ ...	*by* 8, 11 *on* $\{c0, c2\}$,
13.	$\{d1, d2, b0, b2, e1\}$	*by* 8, 11 *on* $\{c1, c2\}$,
14.	$\{e0, e2, a0, a2\}$	*by* 5, 1 *on* $\{e1, a1\}$,
15.	$\{d1, d2, a1, a2\}$	*by* 4, 1 *on* $\{d0, a0\}$,
16.	$\{d1, d2, e0, e2, a2\}$	*by* 14, 15 *on* $\{a0, a1\}$,
17.	$\{d1, d2, b0, b2, a2, e2\}$ ...	*by* 13, 16 *on* $\{e0, e1\}$,
18.	$\{d1, d2, b0, b2, a2\}$	*by* 13, 17 *on* $\{e0, e2\}$,
19.	$\{c0, c1, a0, a1\}$	*by* 3, 1 *on* $\{a2, c2\}$,
20.	$\{d1, d2, b0, b2, c0, a0, a1\}$	*by* 8, 19 *on* $\{c1, c2\}$,
21.	$\{d1, d2, b0, b2, a0, a1\}$ ...	*by* 8, 20 *on* $\{c1, c0\}$,
22.	$\{d1, d2, b0, b2, a0\}$	*by* 18, 21 *on* $\{a2, a1\}$,
23.	$\{d1, d2, b0, b2\}$	*by* 18, 22 *on* $\{a2, a0\}$,
24.	$\{d1, d2, b1, b2\}$	*by* 4, 2 *on* $\{d0, b0\}$,
25.	$\{d1, d2, b2\}$	*by* 23, 24 *on* $\{b0, b1\}$,

Hence we obtain that $\{d1, d2, b2\} \in [\mathcal{A}^\circ]$ where $\mathcal{A}^\circ = \{\{v\} \times \{0, 1, 2\} : v \in \mathcal{V}\}$. Because of symmetry wrt. exchanging 1 with 2, we obtain $\{d1, d2, b1\} \in [\mathcal{A}^\circ]$, hence $\{d1, d2\} \in [\mathcal{A}^\circ]$. Because of the symmetry wrt. exchanging 0 with 2, we get that $\{d0, d1\} \in [\mathcal{A}^\circ]$, hence $\{d0\}$ and $\{d2\}$ are in $[\mathcal{A}^\circ]$. By the resolution on the edge $\{d0, d2\}$, we eventually obtain that $\{\} \in [\mathcal{A}^\circ]$, which proves that $\mathcal{G}$ is not 3-colorable.

3. (n, k)-colorability

We say that a graph $G = (V, E)$ is (n, k)-colorable iff there is $\kappa : \mathcal{V} \to \{0, \dots, n-1\}$ with

$$Card\ (\{b \in V : \{a, b\} \in E, \kappa(a) = \kappa(b)\}) \leq k.$$

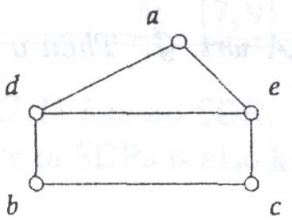
We see that $(n, 0)$-colorability is the usual n-colorability. First we shall deal with $(2, k)$-colorability. We have:

THEOREM. *Let $G = (V, E)$ be a graph and let $\mathcal{G} = (V, \mathcal{E})$ where*

$$\mathcal{E} = \{\{v, v_1, \ldots, v_k\} : \{v_j, v\}, \in E, j = 1, \ldots, k, v_i \neq v_j, i \neq j, i, j = 1, \ldots, k\}.$$

Then G is $(2,k)$-colorable iff $\mathcal{G}$ is 2-colorable, i.e. $\mathcal{E}$ is satisfiable wrt. $\mathcal{G}$.

EXAMPLE *Let $G = (V, E)$ be such that $V = \{a, b, c, d, e\}$ and $E = \{\{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{c. e\}, \{d,e\}\}$.*



In order to decide the $(2,1)$-colorability of G, we must decide the colorability of $\mathcal{G} = (V, \mathcal{E})$, where $\mathcal{E} = \{\{a,b,d\}, \{a,c,e\}, \{a,d,e\}, \{b,c,d\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}\}$. Since $\{b,c,e\}, \{a,c,e\}, \{c,d,e\} \vdash_{\{a,b,d\}} \{c,e\}$, the satisfiability of $\mathcal{E}$ is equivalent to the satisfiability of $\{\{a,b,d\}, \{a,d,e\}, \{b,c,d\}, \{b,d,e\}, \{c,e\}\}$ wrt. $\mathcal{E}$. Since $\{b,c,d\}, \{b,d,e\} \vdash_{\{c,e\}} \{b,d\}$, the duality principle lets us conclude that $(2,1)$-colorability of G reduces to satisfiability of $\mathcal{A}_0$ wrt. $\mathcal{A}_0$, where $\mathcal{A}_0 = \{\{b,d\}, \{c,e\}, \{a,d,e\}\}$. Resolving on this set, however, gives supersets of its elements, only, what one perceives after a closer inspection of the following table:

	bd	ce	ade
bd	x	–	d | ae / b
ce	–	x	e | ad / c
ade	d | b / ae	e | c / ad	x

As it concerns (n,k)-colorability for $n \geq 2$, we have the following:

THEOREM. *Let $G = (V, E)$ be a graph and let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = V \times \{0, \ldots, n-1\}$ and*

$$\mathcal{E} = \{\{(v,i),(v_1,i),\ldots,(v_k,i)\} : \quad i = 0,\ldots,n-1,$$
$$\{v,v_j\} \in E, j = 1,\ldots,k, \ v_j \neq v_l, j \neq l, j,l = 1,\ldots,k\} \ \cup$$
$$\cup \quad \{\{(v,i),(v,j)\} : \ v \in V, i \neq j, i,j = 0,\ldots,n-1\}.$$

Let, moreover, $\mathcal{A} = \{\{v\} \times \{0,\ldots,n-1\} : \ v \in V\}$. *Then* G *is* (n,k)*-colorable iff* $\mathcal{A}$ *is satisfiable with respect to* $\mathcal{G}$.

PROOF.

($\Rightarrow$) *Let* $\kappa : V \to \{0,\ldots,n-1\}$ *be an* (n,k)*-coloring of* G. *Then* κ *itself is consistent and satisfies* $\mathcal{A}$ *wrt.* $\mathcal{G}$.

($\Leftarrow$) *Now let* $\sigma \subseteq \mathcal{V}$ *satisfy* $\mathcal{A}$ *wrt.* $\mathcal{G}$. *Then* σ *is a function and it* (n,k)*-colors* G. ∎

4. Colorability of edges

Let $\mathcal{G} = (\mathcal{V},\mathcal{E})$ be a hypergraph. A function $\kappa : \mathcal{E} \to \{0,\ldots,n-1\}$ is an n-coloring of edges of $\mathcal{G}$ iff no two edges with the same color meet the same vertex. I.e. $Card\ \{\kappa(e) : \ v \in e\} \geq 2$, for $v \in \mathcal{V}$. A hypergraph $\mathcal{G}^\circ = (\mathcal{E},\mathcal{E}^\circ)$, where $\mathcal{E}^\circ = \{\{e \in \mathcal{E} : \ v \in e\} : \ v \in \mathcal{V}\}$ is called the dual hypergraph to the hypergraph $\mathcal{G}$. We easily see that n-colorability of edges of $\mathcal{G}$ is equivalent to usual n-colorability of the dual hypergraph $\mathcal{G}^\circ$ of $\mathcal{G}$.

5. Systems of Distinct Representatives

Let $A_1,\ldots,A_n$ be a family of nonempty finite sets. A System of Distinct representatives, SDR, for $A_1,\ldots,A_n$ is a sequence $\mu_1,\ldots,\mu_n$ of different elements with $\mu_i \in A_i$, $i = 1,\ldots,n$.

THEOREM. *Let* $A_1,\ldots,A_n$ *be as above and let* $\mathcal{G} = (\mathcal{V},\mathcal{E})$ *be such that* $\mathcal{V} = \{(j,a) : \ a \in A_j, j = 1,\ldots,n\}$ *and*

$$\mathcal{E} = \{\{(j,a),(j,b)\} : \ a,b \in A_j, a \neq b\} \cup$$
$$\cup \{\{(j,a),(i,a)\} : \ a \in A_i \cap A_j, i \neq j, i,j = 1,\ldots,n\}\}.$$

Then $A_1,\ldots,A_n$ *has a* SDR *iff* $\mathcal{A}$ *is satisfiable wrt.* $\mathcal{G}$, *where* $\mathcal{A} = \{\{j\} \times A_j : \ j = 1,\ldots,n\}$.

EXAMPLE. *Let* $A = \{2,4\}$, $B = \{1,3,5\}$ *and* $C = D = \{2,4\}$. *In order to decide the existence of* SDR *for* A,B,C,D, *we have to find out weather* $\{\}$ *is in* $[\{\{A2,A4\},\{B1,B3,B5\},\{C2,C4\},\{D2,D4\}\}]_{\mathcal{E}}$, *where* $\mathcal{E}$, *amongst others, contains edges* $\{A2,C2\}$, $\{A2,D2\}$, $\{C2,D2\}$, $\{A4,C4\}$, $\{A4,D4\}$, $\{C4,D4\}$. *We have:*

1.	$\{A2, A4\}$,		
2.	$\{B1, B3, B5\}$,		
3.	$\{C2, C4\}$,		
4.	$\{D2, D4\}$,		
5.	$\{A2, C2\}$	*by* $\{1, 3\}$	*on* $\{A4, C4\}$
6.	$\{D2, C2\}$	*by* $\{3, 4\}$	*on* $\{C4, D4\}$
7.	$\{C2\}$	*by* $\{5, 6\}$	*on* $\{A2, D2\}$
8.	$\{D4\}$	*by* $\{4, 7\}$	*on* $\{C2, D2\}$
9.	$\{A2\}$	*by* $\{1, 8\}$	*on* $\{A4, D4\}$
10.	$\{\}$	*by* $\{7, 9\}$	*on* $\{A2, C2\}$

This proves that A, B, C, D *has no* SDR.

The problem of existence of SDRs is also known as the marriage problem (see [3]).

References

1. Cowen R. H., *Hypergraph satisfiability,* Reports on Mathematical Logic, **24**(1991), 113–118,

2. Cowen R. H., *Combinatorial Analytic Tableaux,* Reports on Mathematical Logic, **27**(1993), 29 – 39,

3. Halmos P. R., Vaughan H. E., *The Marriage Problem,* American Journal of Mathematics, **72**(1949), 214 – 215,

4. Kolany A., *Satisfiability on hypergraphs,* Studia Logica, **52**(1993), 393–404,

5. Linial N., Tarsi M., *Deciding hypergraph 2-colorability by H-resolution,* Theoretical Computer Science, **38**(1985), 343–347.