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NON–LOCAL EQUATIONS IN MATHEMATICS AND PHYSICS. THEORY OF NON–LOCAL ELASTICITY

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The local dependence of a field of one physical quantity (an effect s) at a point $\mathbf{x}$ at time t $(s(x,y,z,t))$ on a field of another physical quantity (a cause p) at the same point $\mathbf{x}$ and at the same time t $(p(x,y,z,t))$ has the following form

$$s(x,y,z,t) = s(p(x,y,z,t)). \tag{1}$$

In a case of time non–locality (materials with memory) the effect s at a point $\mathbf{x}$ at time t depends on the histories of causes at a point $\mathbf{x}$ at all past and present times

$$s(x,y,z,t) = \int_{-\infty}^{t} \alpha(t-t',\epsilon)p(x,y,z,t')\,\mathrm{d}t'. \tag{2}$$

Space non–locality means that the effect s at a point $\mathbf{x}$ at time t depends on causes at all points $\mathbf{x}'$ at time t

$$s(x,y,z,t) = \int_V \beta(|\mathbf{x}-\mathbf{x}'|,\zeta)p(x',y',z',t)\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'. \tag{3}$$

When memory. effect is accompanied by space non–locality we have dependence of the effect s at a point $\mathbf{x}$ at time t on causes at all points $\mathbf{x}'$ and at all times t' prior to and at time t

$$s(x,y,z,t) = \int_{-\infty}^{t}\int_V \gamma(t-t',|\mathbf{x}-\mathbf{x}'|,\epsilon,\zeta)\,p(x',y',z',t')\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\,\mathrm{d}t'. \tag{4}$$

The non–local moduli $\alpha(t-t',\epsilon)$, $\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)$, $\gamma(t-t',|\mathbf{x}-\mathbf{x}'|,\epsilon,\zeta)$ include parameters governing non–locality:

$$\epsilon = \frac{T_i}{T_e} \tag{5}$$

and

$$\zeta = \frac{L_i}{L_e}, \tag{6}$$

where T_i is the internal characteristic time (relaxation time or signal travel time between molecules), T_e is the external characteristic time (the duration of applied force or period of oscillations); L_i is the internal characteristic length (the lattice parameter or granular distance), L_e is the external characteristic length (wave–length or sample thickness) [1].

In the following we shall confine ourselves to a case of space non–locality and describe the properties of the kernel $\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)$ (other kernels $\alpha(t-t',\epsilon)$ and $\gamma(t-t',|\mathbf{x}-\mathbf{x}'|,\epsilon,\zeta)$ have the analogous characteristics):

(*i*) $\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)$ has a maximum at $\mathbf{x}=\mathbf{x}'$.

(*ii*) $\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)$ attenuates rapidly with $|\mathbf{x}-\mathbf{x}'|$ to zero.

(*iii*) β is a continuous function of $|\mathbf{x}-\mathbf{x}'|$ with a bounded support V.

(*iv*) $\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)$ is a delta sequence, and in the classical limit $\zeta\to 0$, β becomes the Dirac delta function

$$\lim_{\zeta\to 0}\beta(|\mathbf{x}-\mathbf{x}'|,\zeta)=\delta(|\mathbf{x}-\mathbf{x}'|).$$

(*v*) For $\zeta\to 1$ non-local theory agrees with atomic lattice dynamics.

(*vi*) $\int_V \beta(|\mathbf{x}-\mathbf{x}'|,\zeta)\ dv(\mathbf{x}')=1.$

Eringen [2] has ascertained the properties of $\beta(|\mathbf{x}'-\mathbf{x}|,\zeta)$ and found several different forms giving a perfect match with the Born-Kármán model of the atomic lattice dynamics and the atomic dispersion curves. For example:

One–dimensional kernels

$$\beta(|x-x'|,\tau)=\begin{cases}\dfrac{1}{\tau L_e}\left(1-\dfrac{|x-x'|}{\tau L_e}\right) & |x-x'|\le\tau L_e,\\[2ex] 0 & |x-x'|\ge\tau L_e,\end{cases} \tag{7}$$

$$\beta(|x-x'|,\tau)=\frac{1}{2\tau L_e}\exp\left(-\frac{|x-x'|}{\tau L_e}\right), \tag{8}$$

$$\beta(|x-x'|,\tau)=\frac{1}{\sqrt{\pi\tau}L_e}\exp\left(-\frac{|x-x'|^2}{\tau L_e^2}\right) \tag{9}$$

with $\tau=k\zeta$ and k being a constant appropriate to each material.

Two–dimensional kernels

$$\beta(|\mathbf{x}-\mathbf{x}'|,\tau)=\frac{1}{\pi\tau L_e^2}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{\tau L_e^2}\right), \tag{10}$$

or

$$\beta(|\mathbf{x}-\mathbf{x}'|,\tau)=\frac{1}{2\pi\tau^2L_e^2}K_0\left(\frac{\sqrt{|\mathbf{x}-\mathbf{x}'|^2}}{\tau L_e}\right), \tag{11}$$

where K_0 is the modified Bessel function.

Three–dimensional kernels

$$\beta(|\mathbf{x}-\mathbf{x}'|,\tau)=\frac{1}{4\pi\tau^2L_e^2\sqrt{|\mathbf{x}-\mathbf{x}'|^2}}\exp\left(-\frac{\sqrt{|\mathbf{x}-\mathbf{x}'|^2}}{\tau L_e}\right), \tag{12}$$

$$\beta(|\mathbf{x}-\mathbf{x}'|,\eta)=\frac{1}{8(\pi\eta)^{3/2}}\exp\left(\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\eta}\right), \tag{13}$$

where $\eta=\frac{1}{4}\tau L_e^2$.

Further, we consider the theory of non–local elasticity which takes into account interatomic long–range forces and use the non–local modulus (13). According to the non–local elasticity the stress at a reference point $\mathbf{x}$ in the body depends not only on the strain at $\mathbf{x}$ but also on the strains at all other points of the body. Several versions of non-local continuum mechanics based on various suggestions have been proposed by Kröner [3], Eringen [4], Edelen [5], Kunin [6] and others.

For the static case with vanishing body force the basic equations for a linear isotropic non-local elastic solid are [4]

$$\nabla\cdot\mathbf{t}=0, \tag{14}$$

$$\mathbf{t}(\mathbf{x})=\int_V\beta(|\mathbf{x}'-\mathbf{x}|,\eta)\,\mathbf{\Sigma}(\mathbf{x}')\;dv(\mathbf{x}'), \tag{15}$$

$$\mathbf{\Sigma}(\mathbf{x}')=\lambda\;\mathrm{tr}\,\mathbf{e}(\mathbf{x}')\,\mathbf{I}+2\mu\mathbf{e}(\mathbf{x}'), \tag{16}$$

$$\mathbf{e}(\mathbf{x}')=\frac{1}{2}[\nabla'\mathbf{u}(\mathbf{x}')+\mathbf{u}(\mathbf{x}')\nabla']. \tag{17}$$

Here $\mathbf{x}$ and $\mathbf{x}'$ are reference and running points, $\mathbf{\Sigma}$ and $\mathbf{t}$ are the local and non-local stress tensors, $\mathbf{u}$ is the displacement vector, $\mathbf{e}$ the linear strain tensor, $\mathbf{I}$ the unit tensor, λ and μ are Lamé constants.

As the non–local modulus (13) is the Green function of the diffusion equation, it is possible to reduce the problem of determining the stress tensor $\mathbf{t}$ to solving the diffusion equation

$$\frac{\partial\mathbf{t}}{\partial\eta}-\nabla^2\mathbf{t}=0 \tag{18}$$

under the initial condition

$$\mathbf{t}\Big|_{\eta=0}=\mathbf{\Sigma} \tag{19}$$

or

$$\nabla^2\bar{\mathbf{t}} - s\bar{\mathbf{t}} = -\mathbf{\Sigma}, \tag{20}$$

where a superposed bar indicates the Laplace transform with respect to η and s is the transform variable.

The above considerations should be modified if we have to solve mixed boundary value problems which are very complicated in non–local elasticity. Then $\mathbf{\Sigma}$ depends on η, and we can use the diffusion equation

$$\nabla^2\bar{\mathbf{t}} - s\bar{\mathbf{t}} = -s\bar{\mathbf{\Sigma}} \tag{21}$$

as an approximate equation for the non–local stress field $\mathbf{t}$. The generalized equation (21) permits to solve those problems of non–local elasticity which cannot be solved using (20).

In axisymmetric case it is possible to obtain a general representation of the solution of non–local elastic problems analogous to the well–known classical representation in terms of biharmonic Love's function [7, 8].

Hereafter, we shall restrict ourselves to the symmetry with respect to the plane $z = 0$ and shall only consider problems with

$$t_{rz}\big|_{z=0} = 0 \qquad \left(\Sigma_{rz}\big|_{z=0} = 0\right) \tag{22}$$

In this case using the Laplace and Hankel transforms, the general representation of components of the non–local stress tensor has the following form

$$t_{zz} = 2\mu \int_0^\infty J_0(r\xi)\xi\, \mathrm{d}\xi \int_0^\eta \beta(\xi, \eta - \tau)\, Q(\xi, |z|, \tau)\, \mathrm{d}\tau,$$

$$t_{rz} = 2\mu \int_0^\infty J_1(r\xi)\xi\, \mathrm{d}\xi \int_0^\eta \beta(\xi, \eta - \tau)\, U(\xi, |z|, \tau)\, \mathrm{d}\tau\, \mathrm{sign} z,$$

$$t_{rr} + t_{\theta\theta} = 2\mu \int_0^\infty J_0(r\xi)\xi\, \mathrm{d}\xi \times$$

$$\times \int_0^\eta \beta(\xi, \eta - \tau)\, [(1+\nu)T(\xi, |z|, \tau) - Q(\xi, |z|, \tau)]\, \mathrm{d}\tau,$$

$$t_{rr} - t_{\theta\theta} = 2\mu \int_0^\infty J_2(r\xi)\xi\, \mathrm{d}\xi \times$$

$$\times \int_0^\eta \beta(\xi, \eta - \tau)[-(1-\nu)T(\xi, |z|, \tau) + Q(\xi, |z|, \tau)]\, \mathrm{d}\tau. \tag{23}$$

Inverse Laplace transforms necessary for obtaining (23) are presented in Appendix.

The function $\beta(\xi, \eta)$ is the inverse Laplace transform

$$\beta(\xi, \eta) = \mathcal{L}^{-1}\left\{sB(\xi, s)\right\}, \tag{24}$$

where $B(\xi, s)$ is an unknown function which should be determined from the boundary conditions, J_n is the Bessel function of the first kind of order n.

The functions $Q(\xi, |z|, t)$, $U(\xi, |z|, t)$, $S(\xi, |z|, t)$, $T(\xi, |z|, t)$ were introduced in [9]. For the sake of convenience we present them here:

$$Q(\xi, |z|, t) = \frac{1}{2}\left(1 - 2t\xi^2\right) T(\xi, |z|, t) - \frac{1}{2}\xi|z| S(\xi, |z|, t) + \frac{2\xi\sqrt{t}}{\sqrt{\pi}} P(\xi, |z|, t),$$

$$U(\xi, |z|, t) = \frac{1}{2}\xi|z| T(\xi, |z|, t) + t\xi^2 S(\xi, |z|, t),$$

$$S(\xi, |z|, t) = \exp(\xi|z|)\,\mathrm{erfc}\left(\xi\sqrt{t} + \frac{|z|}{2\sqrt{t}}\right) - \exp(-\xi|z|)\,\mathrm{erfc}\left(\xi\sqrt{t} - \frac{|z|}{2\sqrt{t}}\right),$$

$$T(\xi, |z|, t) = \exp(\xi|z|)\,\mathrm{erfc}\left(\xi\sqrt{t} + \frac{|z|}{2\sqrt{t}}\right) + \exp(-\xi|z|)\,\mathrm{erfc}\left(\xi\sqrt{t} - \frac{|z|}{2\sqrt{t}}\right)$$

or

$$Q(\xi, |z|, t) = \frac{2\xi^3}{\sqrt{\pi}} \int_t^\infty \tau^{-1/2} (\tau - t) P(\xi, |z|, \tau)\, \mathrm{d}\tau,$$

$$U(\xi, |z|, t) = \frac{\xi^2 |z|}{\sqrt{\pi}} \int_t^\infty \tau^{-3/2} (\tau - t) P(\xi, |z|, \tau)\, \mathrm{d}\tau,$$

$$S(\xi, |z|, t) = \frac{|z|}{\sqrt{\pi}} \int_t^\infty \tau^{-3/2} P(\xi, |z|, \tau)\, \mathrm{d}\tau,$$

$$T(\xi, |z|, t) = \frac{2\xi}{\sqrt{\pi}} \int_t^\infty \tau^{-1/2} P(\xi, |z|, \tau)\, \mathrm{d}\tau \tag{25}$$

with

$$P(\xi, |z|, t) = \exp\left(-\xi^2 t - \frac{z^2}{4t}\right) \tag{26}$$

The functions $Q(\xi,|z|,t)$, $U(\xi,|z|,t)$, $S(\xi,|z|,t)$, $T(\xi,|z|,t)$ given by equations (25) fulfil the following differential

$$\frac{\partial T}{\partial|z|} = \xi S, \qquad \frac{\partial T}{\partial t} = -\frac{2\xi}{\sqrt{\pi t}}P,$$

$$\frac{\partial S}{\partial|z|} = \xi T - \frac{2}{\sqrt{\pi t}}P, \qquad \frac{\partial S}{\partial t} = \frac{|z|}{\sqrt{\pi}t^{3/2}}P,$$

$$\frac{\partial U}{\partial|z|} = \xi(T-Q), \qquad \frac{\partial U}{\partial t} = \xi^2 S,$$

$$\frac{\partial Q}{\partial|z|} = -\xi U, \qquad \frac{\partial Q}{\partial t} = -\xi^2 T$$

and integral relations

$$\int_0^t T(\xi,|z|,\tau)\,\mathrm{d}\tau = -\frac{1}{\xi^2}\left[Q - (1+\xi|z|)\exp(-\xi|z|)\right],$$

$$\int_0^t S(\xi,|z|,\tau)\,\mathrm{d}\tau = \frac{1}{\xi^2}\left[U - \xi|z|\exp(-\xi|z|)\right],$$

$$\int_0^t U(\xi,|z|,\tau)\,\mathrm{d}\tau = \frac{1}{2\xi^2}\left\{t\xi^2 U - \frac{\xi|z|}{2}\left[Q - (1+\xi|z|)\exp(-\xi|z|)\right]\right\},$$

$$\int_0^t Q(\xi,|z|,\tau)\,\mathrm{d}\tau = -\frac{1}{4\xi^2}\Big\{3\left[Q - (1+\xi|z|)\exp(-\xi|z|)\right] - 2t\xi^2(Q-T)$$

$$+\ \xi|z|\left[U - \xi|z|\exp(-\xi|z|)\right]\Big\}$$

which are useful for solving concrete boundary–value problems.

A number of problems solved in the frame–work of non–local theory indicate its power. It manifests some new physical phenomena and overcomes difficulties in classical theory such as classical singularities in stress fields and divergent energies. Using the non–local theory one can obtain more justified results.

REFERENCES

1. A.C.Eringen. Vistas of nonlocal continuum physics. *Int. J. Engng. Sci.*, **30**, 1551–1565 (1992).

2. A.C.Eringen. On differential equations of nonlocal elasticity and solutions of screw dislocations and surface waves. *J. Appl. Phys.*, **54**, 4703–4710 (1983).

3. E.Kröner. Elasticity theory of metals with long range cohesive forces. *Int. J. Solids Structures* **3**, 731-742 (1967).

4. A.C.Eringen. Linear theory of nonlocal elasticity and dispersion of plane waves. *Int. J. Engng. Sci.* **10**, 425-435 (1972).

5. D.G.B.Edelen. Nonlocal field theories. *Continuum Physics* **4**, 75–204 (1976).

6. I.A.Kunin. *Theory of Elastic Media with Microstructure.* Springer, Berlin (1986).

7. I.N.Sneddon. *Fourier Transforms.* McGraw-Hill, New York (1951).

8. W.Nowacki. *Teoria Sprężystości.* PWN, Warszawa (1972).

9. Y.Z.Povstenko. Circular dislocation loops in non–local elasticity. *J. Phys. D: Appl. Phys.* **28**, 105-111 (1995).

APPENDIX

To obtain the representation (23) we need the following formulae

$$\mathcal{L}^{-1}\left\{\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{|z|}{2\sqrt{\pi}t^{3/2}}P,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{1}{2}[S+2\mathrm{e}^{-\xi|z|}],$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{1}{2\xi^2}[U+(2t\xi^2-\xi|z|)\mathrm{e}^{-\xi|z|}],$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{t}{4\xi^2}U - \frac{|z|}{8\xi^3}[Q-(1+\xi|z|)\mathrm{e}^{-\xi|z|}]$$
$$-\frac{t}{2\xi}(|z|-\xi t)\mathrm{e}^{-\xi|z|},$$

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{\xi^2+s}}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{1}{\sqrt{\pi t}}P,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{\xi^2+s}}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{1}{2\xi}[-T+2\mathrm{e}^{-\xi|z|}],$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2\sqrt{\xi^2+s}}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = \frac{1}{2\xi^3}[Q-(1+\xi|z|-2t\xi^2)\mathrm{e}^{-\xi|z|}],$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3\sqrt{\xi^2+s}}\exp\left(-\sqrt{\xi^2+s}|z|\right)\right\} = -\frac{1}{8\xi^5}\{3[Q-(1+\xi|z|)e^{-\xi|z|}]$$

$$+\xi|z|[U-\xi|z|e^{-\xi|z|}]+2t\xi^2(T-Q)+4\xi^2 t(1+\xi|z|-t\xi^2)e^{-\xi|z|}\}.$$