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THE PROBLEM OF DISTRIBUTION OF GOODS

The purpose of this work is the analysis of linear model of economic problem of distribution of goods for the case in which the total demand is higher than the total supply. In the work the problem of transport cost minimization is not considered but the problem is adapted to application of the known solution (see [1]) by the change of the demand.
Changing the demand, the control of real reguirements of market was considered at first and the cost minimization was analysed in the background. The problem of distribution is considered within the established period of time, e.g.: a year.

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In the paper the following denotations will be used:
N – set of natural numbers,
i,(j),(k) – index integer varying from 1 to the fixed $m \in N$, $(n \in N)$, $(h \in N)$, respectively,
$\delta = \delta_j$ – market demand $R = (R_j), \delta_j > 0$,
$\text{б} = \text{б}_i$ – factory supply $F = (F_i), \text{б}_i > 0$,
$w = (w_k)$ – distribution criterion,
e_α – α – dimensional vector of unities,
$(a \cdot b)$ – scalar product of vectors a, b.
We prove now four theorems. It is necessary to introduce first some definitions.

Definition 1. Matrix $A = [a_{jk}]$ is called filtering matrix $\Leftrightarrow \forall_{j,k} a_{jk} \geq 0$ and $e_n A = e_h$.

Corollary. If $a_{jk} \in A$, then $a_{jk} \leq 1$ and $e_n A e_h = h$, because denoting the k-th column of matrix A by $[a_{.k}]$ we have:

$a_{jk} \in A \Rightarrow a_{jk} \geq 0$ and $(e_n \cdot [a_{.k}]) = 1 \Rightarrow a_{jk} \leq 1$.

$e_n A e_h = (e_h \cdot e_h) = h$.

Lemma. (Concering production of filtering matrices).
Assumptions: $A_1, A_2, \ldots, A_s$ – filtering matrices
of $(n \times k)$ dimensions,
$a_{jk}^t \in A_t$ for $t \leqslant s$,

$$\bar{a}_{jk} = \frac{1}{s} \sum_{l=1}^{s} a_{jk}^l.$$

Thesis: $\bar{A} = [\bar{a}_{jk}]$ – filtering matrix.

Proof: $\forall_{j,k,t}\, a_{jk}^t \in A_t \Rightarrow a_{jk}^t \geqslant 0 \Rightarrow \bar{a}_{jk} \geqslant 0.$

$e_n \bar{A} = \frac{1}{s} e_n (A_1 + A_2 + \ldots + A_s) = \frac{1}{s}[s \cdot e_h] = e_h.$

Denoting:
$\delta^\circ = 0$, $\alpha^\circ = (b \cdot e_m)$, $\beta^\circ = h$,
$N^\circ(n) = N(n) = \{1, 2, \ldots, n-1, n\}$ for $n \in N$,
$A = [a_{jk}]$ – filtering matrix,
we will define the following quantities:
$a = (a_j) = Ae_n$, $b^l = (b_j^l)$, $\delta^l = (\delta_j^l)$,

$$b_j^l = \begin{cases} \dfrac{\alpha^{l-1}}{\beta^{l-1}}\, a_j & \text{for } j \in N^{l-1}(n), \\ 0 & \text{for } j \in N(n) \setminus N^{l-1}(n), \end{cases}$$

$$\delta_j^l = A(\delta_j^{l-1}) = \begin{cases} \delta_j & \text{for } j \notin N^l(n), \\ \delta_j^{l-1} + b_j^l & \text{for } j \in N^l(n), \end{cases}$$

$\nu^l = (\delta^l \cdot e_n)$,

$N^l(n) = \{ j \in N^{l-1}(n) : b_j^l + \delta_j^{l-1} \leqslant \delta_j \}$,

$N^{l'}(n) = N^{l-1}(n) \setminus N^l(n)$,

$$a\ e(s) = \sum_{j \in N^s(n)} a_j,$$

$\Delta\delta^l = \delta^l - \delta^{l-1}$,

$$\alpha^l = \begin{cases} (b^l - \Delta\delta^l)e(l'), & \text{if } N^{l'}(n) \neq \emptyset, \\ 0, & \text{if } N^{l'}(n) = \emptyset, \end{cases}$$

$\beta = ae(1)$.

Definition 2. We say that the demand δ is r – times filtered $\Leftrightarrow A(\delta^{r-1}) = \delta$.

Definition 3. We say that the demand δ is fully filtered $\Leftrightarrow \alpha^r = 0$ and $A(\delta^{r-1}) = \delta$.

Solving the problem of distribution we use the following theorems:

Theorem 1. Assumption: the demand δ is l – times filtered.

Thesis: (a) $\alpha^l = b^{l+1} \cdot e(l)$,

(b) $\alpha^l + \nu^l = \alpha^{l-1} + \nu^{l-1}$,

(c) $\alpha^l = (б \cdot e_m) - \nu^l$,

(d) $\nu^l \leq (б \cdot e_m)$,

(e) $\alpha^l = \Leftrightarrow N^{l'}(n) = \emptyset$,

(f) problem of transport (and also distribution) at the demand δ^l has optimum solution.

Proof:

(a) $b^{l+1} e(l) = \frac{\alpha^l}{\beta^l} a_j e(l) = \frac{\alpha^l}{\beta^l} \beta^l = \alpha^l$,

(b) $N^\circ(n) = N^l(n) \cup N^{1'}(n) \cup N^{2'}(n) \cup \ldots \cup N^{(l-1)'}(n) \cup N^{l'}(n)$,

where $N^l(n) \cap N^{\beta'}(n) = \emptyset$ for $\beta' \leq l'$

$N^{\beta_1'}(n) \cap N^{\beta_2'}(n) = \emptyset$ for $\beta_1' \neq \beta_2' \leq l'$,

$(b^l - \Delta\delta^l) \cdot e(l) = 0$,

$(b^l - \Delta\delta^l) \cdot e(\beta') = 0$ for $\beta' < l'$,

$\alpha^l = (b^l - \Delta\delta^l) \cdot e(l') = (b^l - \Delta\delta^l) \cdot e_n = \alpha^{l-1} - \Delta\nu^l \Rightarrow$

$\Rightarrow \alpha^l + \nu^l = \alpha^{l-1} + \nu^{l-1}$,

(c) $\alpha^\circ + \nu^\circ = (б \cdot e_m)$ and $\alpha^\circ + \nu^\circ = \alpha^l + \nu^l = (б \cdot e_m) \Rightarrow \alpha^l = (б \cdot e_m) - \nu^l$,

(d) $\alpha^l + \nu^l = (б \cdot e_m)$ and $\alpha^l \geq 0 \Rightarrow \nu^l \leq (б \cdot e_m)$,

(e) $\Leftarrow$) from definitoin,

$\Rightarrow$) $\alpha^l = 0 \Leftrightarrow (b^l - \Delta\delta^l) = 0 \Leftrightarrow \forall_j \delta_j^l = \delta_j^{l-1} + b_j^l \Leftrightarrow$

$\Leftrightarrow j \in N^l(n) \Leftrightarrow j \notin N^{l'}(n) \Leftrightarrow N^{l'}(n) = \emptyset$,

(f) from (d)$\Rightarrow (\delta^l \cdot e_m) \leq (б \cdot e_m)$, from ref. [1] exercise 1, page 44 there is a permissible solution, there is finite number of such solutions and then the optimum solution exists.

Theorem 2. If the demand δ is fully filtered, then $\exists$ $l \in N$ that the following conditions

(a) $\alpha^l = 0$,

(b) $\nu^l = (e_m \cdot б)$,

(c) $N^{l'}(n) = \emptyset$,

(d) $b^{l+1} = 0$ are equivalent.

Proof: from theorem 1 it results that:

$(a)\Leftrightarrow(d)\exists l:\alpha^l=0\Leftrightarrow b^{l+1}\cdot e(l)=0\Leftrightarrow b^{l+1}=0,$

$(a)\Leftrightarrow(c)\exists l:\alpha^l=0\Leftrightarrow N^{l'}(n)=\emptyset,$

$(a)\Leftrightarrow(b)\exists l:\nu^l=(\text{б}\cdot e_m)\Leftrightarrow\alpha^l=0.$

Theorem 3. If the demand δ is fully filtered then $\exists l\leqslant n:\alpha^l=0$.

Proof: assume that $\delta^1<\delta\Rightarrow\forall_{j\in N}\ \delta^1_j<\delta_j\Rightarrow N^1(n)=N(n)\Rightarrow$

$\Rightarrow N^{1'}(n)=\emptyset\Rightarrow\alpha^1=0.$

If $\exists j_1\in N(n):\delta^1_{j_1}=\delta_{j_1}\Rightarrow N^{1'}(n)=N(n)\setminus\{j_1\}$.

If $\delta^2_j<\delta_j\ \forall_j\in N^{1'}(n)\Rightarrow N^{2'}(n)=\emptyset\Rightarrow\alpha^2=0.$

Let $\exists j_2\in N^2(n):\delta^2_{j_2}=\delta_{j_2}\Rightarrow j_1,j_2\notin N^{2'}(n)\Rightarrow$ the set $N^{2'}(n)$ can contain (n-2) elements at most.

Then, after at most n – times filtration of the demand δ the set $N^{n'}(n)=\emptyset\Rightarrow\alpha^n=0\Rightarrow\exists l\leqslant n:\alpha^l=0.$

Theorem 4. (Principial theorem concering distribution).

The problem of transport (and hence, distribution, as presented above) has always the optimum solution taking full advantages of productivity of the factory F.

Proof. If $(\text{б}\cdot e_m)\geqslant(\delta\cdot e_n)$, then see ref. [1].

If $(\text{б}\cdot e_m)<(\delta\cdot e_n)$, then from theorem 1(f) and theorem3 the thesis results.

REFERENCES

[1] D. Gale, Theory of economic linear models, PWN, Warszawa 1969, (in Polish).

STRESZCZENIE

W artykule udowodniono twierdzenie (twierdzenie 4) rozszerzające wynik D. Galleá o przypadek $(\text{б}\cdot e_m)<(\delta\cdot e_n)$ dla wektorów popytu i podaży o elementach dodatnich. Twierdzenie to może mieć zastosowanie szersze na przykład w ekonomicznych zagadnieniach allokacji.