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Free algebras over some varieties

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0.

Let $\tau : F \to N$ be a fixed type of algebras, where F is the set of fundamental operation symbols and N is the set of non-negative integers. If φ is a term of type τ we denote by $F(\varphi)$ the set of all fundamental operation symbols occuring in φ and by $Var(\varphi)$ the set of all variables occuring in φ. If φ is a term then writting $\varphi(x_{i_0}, \ldots, x_{i_{m-1}})$ we shall mean that $Var(\varphi) = \{x_{i_0}, \ldots, x_{i_{m-1}}\}$. Let F_0 be a subset of F.

Definition 1 (see [3]) *An identity $\varphi_1 = \varphi_2$ of type τ will be called F_0-regular iff $F(\varphi_1) \subset F_0$, $F(\varphi_2) \subset F_0$, and $Var(\varphi_1) = Var(\varphi_2)$.*

Definition 2 (see [3]) *An identity $\varphi_1 = \varphi_2$ of type τ will be called F_0-symmetrical iff $F(\varphi_1) \cap F_0 \neq \emptyset$ and $F(\varphi_2) \cap F_0 \neq \emptyset$.*

If A is a set we denote by $Fin(A)$ the set of all non–void finite subsets of A. Let $F_1, F_2 \subset F$ and $F_1 \cup F_2 = F$, $F_1 \cap F_2 = \emptyset$ and $\{f \in F : \tau(f) = 0\} \subset F_2$. Let $\mathbf{K}$ be a variety of algebras of type τ. We denote by $\mathrm{Id}(\mathbf{K})$ the set of all identities of type τ satisfied in each member of $\mathbf{K}$. Denote by $\mathbf{K}_{F_1}^{F_2}$ the variety of type τ defined by all F_1–regular identities and all F_2–symmetrical identities belonging to $\mathrm{Id}(\mathbf{K})$, $\mathbf{K}_{F_1}$– the variety of type $\tau \mid F_1$ defined by all identities of type $\tau \mid F_1$ belonging to $\mathrm{Id}(\mathbf{K})$.

In [3] we defined a construction $\mathcal{S}(\mathcal{A})$ called the sum of an upper (F_1, F_2)–semilattice ordered system $\mathcal{A}$ of algebras. In [3] we showed that if

(c) an identity $\varphi(x, y) = x$ belongs to $\mathrm{Id}(\mathbf{K})$, where $F(\varphi(x, y)) \subset F_1$, then an algebra $\mathfrak{A}$ belongs to $\mathbf{K}_{F_1}^{F_2}$ iff $\mathfrak{A}$ is the sum of an upper (F_1, F_2)–semilattice ordered system of algebras from $\mathbf{K}_{F_1}$ and some algebra from $\mathbf{K}$.

In this paper we show that if $\mathbf{K}$ is nondegenerated and satisfies (c) then an algebra is a free algebra over $\mathbf{K}_{F_1}^{F_2}$ iff it is the sum of an upper (F_1, F_2)–semilattice ordered system of free algebras over $\mathbf{K}_{F_1}$ and some free algebra over $\mathbf{K}$. The case $F_2 = \emptyset$ was considered in [2].

1.

In [3] we defined the following system and construction.

Definition 3 *A quadruple*

(1) $$\mathcal{A} =< (F_1, F_2), (I, \leq), \{\mathfrak{A}_i\}_{i\in I}, \{h_i^j\}_{i,j\in I, i\leq j} >$$

will be called an upper (F_1, F_2)–semilattice ordered system of algebras if it satisfies the following conditons:

(i) $F_1 \cup F_2 = F,\ F_1 \cap F_2 = \emptyset\ ,\ \{f \in F : \tau(f) = 0\} \subset F_2\ .$

(ii) $(I, \leq)$ *is join-semilattice; if* $F_2 \neq \emptyset$ *then* $(I, \leq)$ *has the greatest element* u.

(iii) $\mathfrak{A}_u$ *is an algebra of type* τ *and* $\mathfrak{A}_u = (A_u; F^u)$*; for every* $i \in I,\ i \neq u$, $\mathfrak{A}_i$ *is an algebra of type* $\tau \mid F_1$ *and* $\mathfrak{A}_i = (A_i; F_1^i)$*, where* $A_i \cap A_j = \emptyset$ *if* $i \neq j$.

(iv) The set $\{h_i^j\}_{i,j\in I, i\leq j}$ *satisfies the following:*

(a1) for every $i, j \in I,\ i \leq j,\ h_i^j$ *is a mapping of* A_i *into* A_j;

(a2) for every $i \in I,\ h_i^i$ *is an identity map on* A_i;

(a3) for every $i, j \in I$ *such that* $i \leq j \neq u,\ h_i^j$ *is a homomorphism of* $\mathfrak{A}_i$ *into* $\mathfrak{A}_j$;

(a4) for every $i \in I,\ i \neq u,\ h_i^u$ *is a homomorphism of* $\mathfrak{A}_i$ *into the reduct* (A_u, F_1^u) *of* $\mathfrak{A}_u$;

(a5) for every $i, j, k \in I$ *such that* $i \leq j \leq k$ *we have* $h_j^k \circ h_i^j = h_i^k$.

For an upper (F_1, F_2)–system $\mathcal{A}$ of algebras we define an algebra $\mathcal{S}(\mathcal{A})$ of type τ as follows

$$\mathcal{S}(\mathcal{A}) = (\bigcup_{i\in I} A_i; F^{\mathcal{S}}),$$

where for $f \in F$, $a_j \in A_{i_j}$, $j = 0, \ldots, \tau(f)-1$ the operation f^S is defined by the formula:

$$f^S(a_0,\ldots,a_{\tau(f)-1}) = \begin{cases} f^k(h^k_{i_0}(a_0),\ldots,h^k_{i_{\tau(f)-1}}(a_{\tau(f)-1})), & \text{for } f\in F_1 \text{ and} \\ & k=\sup\{i_0,\ldots,i_{\tau(f)-1}\} \\ f^u(h^u_{i_0}(a_0),\ldots,h^u_{i_{\tau(f)-1}}(a_{\tau(f)-1})), & \text{for } f\in F_2. \end{cases}$$

The algebra $\mathcal{S}(\mathcal{A})$ is called the sum of the upper (F_1, F_2)–semilattice ordered system $\mathcal{A}$ of algebras.
In [3] we proved the following:

Theorem 1 *If a variety* $\mathbf{K}$ *of type* τ *satisfies (c) then an algebra* $\mathfrak{A}$ *belongs to* $\mathbf{K}^{F_2}_{F_1}$ *iff* $\mathfrak{A}$ *is the sum of an upper* (F_1,F_2)*–semilattice ordered system of algebras* $\{\mathfrak{A}_i\}_{i\in I}$*, where* $\mathfrak{A}_i \in \mathbf{K}_{F_1}$ *for* $i \neq u$ *and* $\mathfrak{A}_u \in \mathbf{K}$.

2.

Let $\mathbf{K}$ be a nondegenerated variety of type τ satisfying (c) and $\mathcal{W}(G) = (W(G); F^{\mathcal{W}(G)})$ be a free algebra over $\mathbf{K}^{F_2}_{F_1}$ with the set G of free generators. We denote by $\circ$ the binary function induced in $\mathcal{W}(G)$ by a fixed term $\varphi(x,y)$ from (c) (see [3]). Let $g_1,\ldots,g_n \in G$ where $g_i \neq g_j$ for $i \neq j$; $i,j = 1,\ldots,n$ and $\mathcal{C}(\{g_1,\ldots,g_n\}) = (C(\{g_1,\ldots,g_n\}); F_1^{\mathcal{C}})$ be a subalgebra of the reduct $(W(G); F_1^{\mathcal{W}(G)})$ of $\mathcal{W}(G)$ generated by the set $\{c_1,\ldots,c_n\}$, with

(2) $$c_k = g_k \circ g_1 \circ \ldots \circ g_{k-1} \circ g_{k+1} \circ \ldots \circ g_n \text{ for } k = 1,\ldots,n.$$

Lemma 1 $\mathcal{C}(\{g_1,\ldots,g_n\})$ *is a free algebra over* $\mathbf{K}_{F_1}$ *with the set* $c_1,\ldots,c_n$ *of free generators.*

Proof.
Let

(3) $$\varphi_1(x_{j_1},\ldots,x_{j_m}) = \varphi_2(x_{j_{m+1}},\ldots,x_{j_{m+s}})$$

belongs to $\mathrm{Id}(\mathbf{K}_{F_1})$ and $a_1,\ldots,a_m,a_{m+1},\ldots,a_{m+s} \in C(\{g_1,\ldots,g_n\})$. We procedee analogously as in Lemma 1 from [2], namely: Each a_j can be expressed by generators c_k and because of (2), by $g_1,\ldots,g_n$. Let

(4) $$a_j = \psi_j^{\mathcal{W}(G)}(g_1,\ldots,g_n),$$

where $F(\psi_j) \subset F_1$, each of $g_1, \ldots, g_n$ actually occurs for $j = 1, \ldots, m, m+1, \ldots, m+s$. Since the identity

$$\begin{aligned}(5) \qquad & \varphi_1(\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)) = \\ & = \varphi_2(\psi_{m+1}(x_1, \ldots, x_n), \ldots, \psi_{m+s}(x_1, \ldots, x_n))\end{aligned}$$

is F_1–regular and belongs to Id(**K**), so it is satisfied in $\mathcal{W}(G)$. Thus by (4) and (5) $\varphi_1^{\mathcal{C}}(a_1, \ldots, a_m) = \varphi_2^{\mathcal{C}}(a_{m+1}, \ldots, a_{m+s})$ and $\mathcal{C}(\{g_1, \ldots, g_n\}) \in \mathbf{K}_{F_1}$.

We observe that $c_1, \ldots, c_n$ are different. Obviously $\{c_1, \ldots, c_n\}$ is a set of generators of $\mathcal{C}(\{g_1, \ldots, g_n\})$. We shall prove that if an identity $\varphi_1 = \varphi_2$ holds on $\{c_1, \ldots, c_n\}$ then $(\varphi_1 = \varphi_2) \in \mathrm{Id}(\mathbf{K}_{F_1})$.

Let $\varphi_1^{\mathcal{C}}(c_{j_1}, \ldots, c_{j_m}) = \varphi_2^{\mathcal{C}}(c_{j_{m+1}}, \ldots, c_{j_{m+s}})$ and $c_{j_k} \in \{c_1, \ldots, c_n\}$ where $k = 1, \ldots, m, m+1, \ldots, m+s$. Denote by $\varphi(x, y) = x \circ y$. Then
$\varphi_1^{\mathcal{W}(G)}(g_{j_1} \circ \ldots \circ g_{j_1-1} \circ g_{j_1+1} \circ \ldots \circ g_n, \ldots, g_{j_m} \circ \ldots \circ g_{j_m-1} \circ g_{j_m+1} \circ \ldots \circ g_n) =$
$\varphi_2^{\mathcal{W}(G)}(g_{j_{m+1}} \circ \ldots \circ g_{j_{m+1}-1} \circ g_{j_{m+1}+1} \circ \cdots \circ g_n, \ldots, g_{j_{m+s}} \circ \cdots \circ g_{j_{m+s}-1} \circ g_{j_{m+s}+1} \circ \cdots \circ g_n)$
hence the identity
$\varphi_1(x_{j_1} \circ \ldots \circ x_{j_1-1} \circ x_{j_1+1} \circ \ldots \circ x_n, \ldots, x_{j_m} \circ \ldots \circ x_{j_m-1} \circ x_{j_m+1} \circ \ldots \circ x_n) =$
$\varphi_2(x_{j_{m+1}} \circ \ldots \circ x_{j_{m+1}-1} \circ x_{j_{m+1}+1} \circ \ldots \circ x_n, \ldots, x_{j_{m+s}} \circ \ldots \circ x_{j_{m+s}-1} \circ x_{j_{m+s}+1} \circ \ldots \circ x_n)$ is F_1–regular and belongs to $\mathrm{Id}(\mathbf{K}_{F_1}^{F_2})$. So it belongs to $\mathrm{Id}(\mathbf{K}_{F_1})$. Thus by (c) the identity $\varphi_1(x_{j_1}, \ldots, x_{j_m}) = \varphi_2(x_{j_{m+1}}, \ldots, x_{j_{m+s}})$ belongs to $\mathrm{Id}(\mathbf{K}_{F_1})$.

Let $F_2 \neq \emptyset$, $f_0 \in F_2$ and $b_0, \ldots, b_{\tau(f_0)-1} \in W(G)$. Let
$G' = \{\varphi^{\mathcal{W}(G)}(g, f_0^{\mathcal{W}(G)}(b_0, \ldots, b_{\tau(f_0)-1}))\}_{g \in G}$. Denote by $\mathcal{P}(G')$ the subalgebra of $\mathcal{W}(G)$ generated by the set G' and $\mathcal{P}(G') = (P(G'); F^{\mathcal{P}(G')})$.

Lemma 2 *$\mathcal{P}(G')$ is a free algebra over* **K** *with the set G' of free generators.*

Proof. Let

$$(6) \qquad \varphi_1(x_{j_0}, \ldots, x_{j_{m-1}}) = \varphi_2(x_{j_m}, \ldots, x_{j_{m+s-1}})$$

belongs to Id(**K**) and $a_0, \ldots, a_{m-1}, a_m, \ldots, a_{m+s-1} \in P(G')$ so for $j = 1, \ldots, m-1, m, \ldots, m+s-1$ we have

$$\begin{aligned}(7) \qquad a_j = \psi_j^{\mathcal{W}(G)}(&\varphi^{\mathcal{W}(G)}(g_0^j, f_0^{\mathcal{W}(G)}(b_0, \ldots, b_{\tau(f_0)-1})), \ldots, \\ &\varphi^{\mathcal{W}(G)}(g_{n_j-1}^j, f_0^{\mathcal{W}(G)}(b_0, \ldots, b_{\tau(f_0)-1})))\end{aligned}$$

where $g_s^j \in G$. But

$$\begin{aligned}
&\varphi_1^{\mathcal{W}(G)}(\psi_0^{\mathcal{W}(G)}(\varphi^{\mathcal{W}(G)}(g_0^0, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1})),\ldots,\\
&\varphi^{\mathcal{W}(G)}(g_{n_0-1}^0, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1}))),\ldots,\\
&\psi_{m-1}^{\mathcal{W}(G)}(\varphi^{\mathcal{W}(G)}(g_0^{m-1}, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1})),\ldots,\\
&\varphi^{\mathcal{W}(G)}(g_{n_{m-1}-1}^{m-1}, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1})))) =\\
(8)\quad &= \varphi_2^{\mathcal{W}(G)}(\psi_m^{\mathcal{W}(G)}(\varphi^{\mathcal{W}(G)}(g_0^m, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1})),\ldots,\\
&\varphi^{\mathcal{W}(G)}(g_{n_m-1}^m, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1}))),\ldots,\\
&\psi_{m+s-1}^{\mathcal{W}(G)}(\varphi^{\mathcal{W}(G)}(g_0^{m+s-1}, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1})),\ldots,\\
&\varphi^{\mathcal{W}(G)}(g_{n_{m+s-1}-1}^{m+s-1}, f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1}))))
\end{aligned}$$

since the identity

$$\begin{aligned}
&\varphi_1(\psi_0(\varphi(x_0^0, f_0(y_0,\ldots,y_{\tau(f_0)-1})),\ldots,\varphi(x_{n_0-1}^0, f_0(y_0,\ldots,y_{\tau(f_0)-1}))),\ldots,\\
&\psi_{m-1}(\varphi(x_0^{m-1}, f_0(y_0,\ldots,y_{\tau(f_0)-1})),\ldots,\varphi(x_{n_{m-1}-1}^{m-1}, f_0(y_0,\ldots,y_{\tau(f_0)-1})))) =\\
&= \varphi_2(\psi_m(\varphi(x_0^m, f_0(y_0,\ldots,y_{\tau(f_0)-1})),\ldots,\varphi(x_{n_m-1}^m, f_0(y_0,\ldots,y_{\tau(f_0)-1}))),\ldots,\\
&\psi_{m+s-1}(\varphi(x_0^{m+s-1}, f_0(y_0,\ldots,y_{\tau(f_0)-1})),\ldots,\\
&\varphi(x_{n_{m+s-1}-1}^{m+s-1}, f_0(y_0,\ldots,y_{\tau(f_0)-1}))))
\end{aligned}$$

is F_2–symmetrical and belogs to $\mathrm{Id}(\mathbf{K})$. Thus by (7), (8)

$$\varphi_1^{\mathcal{C}(G')}(a_0,\ldots,a_{m-1}) = \varphi_2^{\mathcal{C}(G')}(a_m,\ldots,a_{m+s-1})$$

and $\mathcal{P}(G')$ belongs to $\mathrm{Id}(\mathbf{K})$.

Let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ belongs to $\mathbf{K}$ and h_1 is a mapping $h_1 : G' \to A$. Take a mapping $h_2 : G \to A$ such that $h_2(a) = h_1(\varphi^{\mathcal{W}(G)}(a, f_0(b_0,\ldots,b_{\tau(f_0)-1})))$ for $a \in G$. $\mathcal{W}(G)$ is a free over $\mathbf{K}_{F_1}^{F_2}$, so h_2 can be extended to a homomorphism h of $\mathcal{W}(G)$ into $\mathfrak{A}$. Since $x \circ y = x$ is satisfied in $\mathfrak{A}_u$ so

(9)

$$h(\varphi^{\mathcal{W}(G)}(a, f_0(b_0,\ldots,b_{\tau(f_0)-1}))) = \varphi^{\mathfrak{A}}(h(a), h(f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1}))).$$

Since $\mathfrak{A} \in \mathbf{K}$, so

$$\begin{aligned}
(10)\quad &\varphi^{\mathfrak{A}}(h(a), h(f_0^{\mathcal{W}(G)}(b_0,\ldots,b_{\tau(f_0)-1}))) = h(a) = h_2(a) =\\
&= h_1(\varphi^{\mathcal{W}(G)}(a, f_0(b_0,\ldots,b_{\tau(f_0)-1}))).
\end{aligned}$$

Thus by (9), (10) $h \mid P(G')$ is a homomorphism being an extension of h_1.

Theorem 2 *If $\mathbf{K}$ is a nondegenerated variety satisfying (c) then an algebra $\mathfrak{A}(G) = (A(G); F^{\mathfrak{A}(G)})$ is a free algebra over variety $\mathbf{K}_{F_1}^{F_2}$ with the set G of free generators iff $\mathfrak{A}(G) = S(\mathcal{A})$, where $\mathcal{A}$ is an upper (F_1, F_2)–semilattice ordered system of algebras satisfying (1) and*

(1^G) $I = Fin(G) \cup \{G\}$, $u = G$, $\leq = \subset$.

(2^G) *$\mathfrak{A}_i$ for $i \neq G$ is a free algebra over variety $\mathbf{K}_{F_1}$ with the set $\{a_{(k,i)}\}_{k \in i}$ of free generators, $\mathfrak{A}_u$ is a free algebra over variety $\mathbf{K}$ with the set $\{a_{(k,G)}\}_{k \in G}$ of free generators.*

(3^G) *For $i \leq j$, $i, j \in I$, $h_i^j(a_{(k,i)}) = a_{(k,j)}$.*

(4^G) *For $i \leq j$, $i, j \in I$, h_i^j is the embeding of $\mathfrak{A}_i$ into $\mathfrak{A}_j^*$, where $\mathfrak{A}_j^* = \mathfrak{A}_j$ if $j \neq G$ and $\mathfrak{A}_G^* = (A(G); F_1^{\mathfrak{A}})$.*

Proof ($\Rightarrow$). The proof is based on Theorem 1 from section 1. Consider a relation R in $\mathfrak{A}(G)$ defined by

$$aRb \text{ iff } \varphi^{\mathfrak{A}(G)}(a,b) = a \text{ and } \varphi^{\mathfrak{A}(G)}(b,a) = b.$$

Using Theorem 4 (see [3]) one can prove that $\mathfrak{A}(G)$ is the sum of an upper (F_1, F_2)–semilattice ordered system of the form (1), where elements a, b belongs to the same A_i iff aRb. Let $g_1, \ldots, g_n \in A(G)$ and $g_i \neq g_j$ for $i \neq j$, $i, j = 1, \ldots, n$. To use Lemma 1 from section 1 we put $c_k = g_k \circ g_1 \circ \ldots \circ g_{k-1} \circ g_{k+1} \circ \ldots \circ g_n$. Using R we can show that every $\mathfrak{A}_i$ for $i \neq u$ is generated by $\{\varphi^{\mathfrak{A}(G)}(g, f_0^{\mathfrak{A}(G)}(b_0, \ldots, b_{\tau(f_0)-1}))\}_{g \in G}$ where f_0 is some operation symbol belonging to F_2. By Lemma 1 $\mathfrak{A}_i$ is free over $\mathbf{K}_{F_1}$ and by Lemma 2 $\mathfrak{A}_u$ is free over $\mathbf{K}$. Since every $\mathfrak{A}_i$ is generated by some $c_{j_1}, \ldots, c_{j_{n_i}}$, where c_{j_s} is generated by some $g_{j_1}, \ldots, g_{j_{n_i}}$ for $s = 1, \ldots, n_i$. We can put $i = \{g_{j_1}, \ldots, g_{j_{n_i}}\}$, put $u = G$. Then we can substitute $\leq$ by $\subset$. In [3] h_i^j was defined as follows: for $x \in A_i$, $b \in A_j$ $h_i^j(x) = x \circ b$. Thus conditions (3^G) and (4^G) are also satisfied.

Proof ($\Leftarrow$). The fact that $\mathcal{S}(\mathcal{A}) \in \mathbf{K}_{F_1}^{F_2}$ follows from Theorem 1. The fact that $\{a_{(i,\{i\})}\}_{\{i\} \in I}$ is a set of generators of $\mathcal{S}(\mathcal{A})$ follows from the formula $a_{(k,j)} = a_{(k,\{k\})} \circ a_{(i_0,\{i_0\})} \circ \ldots \circ a_{(i_{n-1},\{i_{n-1}\})}$ for $j = \{k, i_0, \ldots, i_{n-1}\}$ and the formula $h_{\{i\}}^G(a_{(i,\{i\})}) = a_{(i,\{i\})} \circ b$ where $b = \psi^{\mathfrak{A}(G)}(a_{(i,\{i\})}, \ldots, a_{(i,\{i\})})$ for an arbitrary term ψ with $F(\psi) \cap F_2 \neq \emptyset$. The fact that every element $a \in \mathcal{S}(\mathcal{A})$ has the unique reprezentation on the set $\{a_{(i,\{i\})}\}_{\{i\} \in I}$ (as a realization of some term) follows from the definition $\mathcal{S}(\mathcal{A})$.

Literatura

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