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Extensions of the Grzegorczyk logic determined by some countable Boolean algebras

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Abstract

It is shown that a chain of type of $\omega+1$ of modal logics: $\mathbf{Tr} = \mathbf{Grz} + \mathbf{B}_1 \supset \mathbf{Grz} + \mathbf{B}_2 \supset \mathbf{Grz} + \mathbf{B}_3 \supset \ldots \supset \mathbf{Grz}$ (all of them are extensions of the Grzegorczyk logic **Grz**), contains all and only such modal logics which can be obtained as sets of formulae that are valid in the Stone spaces of countable superatomic Boolean algebras. Some topological conditions which correspond to the Grzegorczyk logic are presented.

Let X be a topological space. $E(X)$ denotes a set of formulae from the set L_M of all formulae of propositional modal logic (with connectives $\rightarrow, \wedge, \vee, \neg, \Box, \Diamond$) that are valid in topological Boolean algebra, TBA, 2^X. Hence $E(X)$ is a set of formulae that have value X for every valuation of propositional variables with subsets of X. Necessity functor $\Box$ is interpreted as a topological interior operation *int* (possibility, $\Diamond$, as a topological closure operation *cl*); the remaining connectives are interpreted in a standard way (cf. e.g., [16], [17]).

Every Boolean algebra **A** has it's corresponding Stone space **St A** of ultrafilters on **A**. The Stone topology of **St A** is determined bya subbase of sets of the form $s(a) = \{F \in \mathbf{St\ A} : a \in F\}$, for $a \in A$; hence an interior operation *int* is defined.

It is known that, for any topological space X, $\mathbf{S4} \subseteq E(X)$, where **S4** is a set of theorems of the logic **S4** of Lewis (J.C.C. McKinsey, A.Tarski [11], [12]). For any discrete space $X, E(X) = \mathbf{Tr}$, where **Tr** is so called *trivial* *logic* i.e. the classical logic plus $(\Box p \leftrightarrow p)$. So we have $\mathbf{S4} \subseteq E(\mathbf{St\ A}) \subseteq \mathbf{Tr}$, for any nontrivial Boolean algebra $\mathbf{A}$.

The questions arise: What modal logics (extensions of S4) are determined by Stone spaces of Boolean algebras? What properties of Boolean algebras - or of their Stone spaces - are reflected by the corresponding logics? Here we give the answer to these questions in case of countable and superatomic Boolean algebras. We show that only the logics $\mathbf{Grz} + \mathbf{B}_n$, for $n \in N$, of extensions of the Grzegorczyk logic **Grz** can be obtained as $E(\mathbf{St\ A})$, for a countable superatomic Boolean algebra $\mathbf{A}$, and that all of these logics can be obtained in such a way. Moreover, the logic $\mathbf{Grz} + \mathbf{B}_{n+1}$ is obtained in such a way iff the n-th Cantor-Bendixson derivative of $\mathbf{A}$ is finite (Corollary 8). We also show (in Corollary 2) for every countable Boolean algebra $\mathbf{A}$ that the Grzegorczyk axiom is valid in $\mathbf{St\ A}$ iff $\mathbf{A}$ is superatomic.

In the paper we will use standard notions of topology (cf. K.Kuratowski [9], A. Engelking [5]), as well as Boolean algebras and Stone duality (cf. D. Monk [14]); we will also make use of results from the book of H. Rasiowa and R. Sikorski [16].

A. Spaces with a dense-in-itself set - non-atomic algebras.

C.C. McKinsey and A.Tarski ([11], see also [16]) proved, that for any dense-in-itself metric space X, $E(X) = \mathbf{S4}$. From this it follows

Theorem 1. *If* $\mathbf{A}$ *is a countable Boolean algebra, which is not atomic, then*

$$E(\mathbf{St\ A}) = \mathbf{S4}.$$

Proof. From countability of $\mathbf{A}$ it follows that $\mathbf{St\ A}$ is a metric space (see e.g. [14]). Since $\mathbf{A}$ is not atomic, there is $a \in A$, such that $\{x \in A : x \leq a\}$ does not contain any atom; then there is an open dense-in-itself set $U \subseteq \mathbf{St\ A}$. A mapping f defined by $f(Y) = Y \cap U$, for $Y \subseteq \mathbf{St\ A}$, is a homomorphism of the TBA 2^{StA} onto the TBA 2^{U}, (see [16], 3.1. p.98), and hence $E(\mathbf{St\ A}) \subseteq E(U) = \mathbf{S4}$. q.e.d.

Corollary 1. $E(\text{St } \mathbf{A}_\omega) = \mathbf{S4}$, *where* $\mathbf{A}_\omega$ *is the* ω*-generated free Boolean algebra.*

B. Some topological conditions corresponding to the Grzegorczyk logic.

Grzegorczyk normal modal logic **Grz** is an extension of S4 Lewis logic with the formula Grz :

$$\Box(\Box(p \to \Box p) \to p) \to p.$$

i.e. $\mathbf{Grz} = \mathbf{S4} + Grz$. **Grz** was introduced by A.Grzegorczyk [7] (see the proof of Thm.3).

It can be verified that the same set **Grz** can be obtained, if the logic S4 with the rules Modus Ponens and necessitation $\varphi/\Box\varphi$ is extended with a rule R_{grz} defined by: $\Box(\varphi \to \Box\varphi) \to \varphi/\varphi$. The rule R_{grz} preserves validity in topological Boolean algebras. It can be checked that a TBA **B** is a model of **Grz** iff for any valuation ν in B:

$$\nu(\Box(\varphi \to \Box\varphi) \to \varphi) = 1 \quad \text{implies that} \quad \nu(\varphi) = 1,$$

or, equivalently,

$$x \neq 0 \quad \text{implies that} \quad x - cl(clx - x) \neq 0, \quad \text{for any element } x \text{ of } B.$$

Observe also that for any X, $Grz \in E(X)$ iff

$$(*) \qquad A \subseteq cl(A - cl(clA - A)), \quad \text{for every} \quad A \subseteq X.$$

Residue in the sense of Hausdorff. F. Hausdorff [8] introduced a notion of residue of a set res $A = cl(clA - A) - (clA - A)$ or, equivalently, res $A = A \cap cl(clA - A)$.

Observe that: res $A \subseteq A$, and res A is a closed set in A ([9], p.101).

Residue of transfinite order is defined by induction on ordinals

$$A = \text{res}^0 A, \quad \text{res}^{\alpha+1} A = \text{res}^\alpha A, \quad \text{res}^\lambda A = \bigcap\{\text{res}^\alpha A : \alpha < \lambda\}$$

for a limit ordinal.

Since the sequence of residues is decreasing, there exists an ordinal β such that $\text{res}^{\beta+1}A = \text{res}^{\beta}A$; such $\text{res}^{\beta}A$ is called „the last" residue.

A set A is called *resolvable into an alternated series* of decreasing closed sets F_α, if

$$A = (F_1 - F_2) \cup (F_3 - F_4) \cup (F_5 - F_6) \cup \ldots \cup (F_\alpha - F_{\alpha+1}) \cup \ldots,$$

(cf. K. Kuratowski [9], Ch.I, par.13, pp. 96 -102)

The last residue $\text{res}^{\beta}A$ is empty iff A is resolvable into an alternated series of decreasing closed sets (cf. [8] and [9], p.103).

Observe also that $A - \text{res}\, A = A - (A \cap cl(clA - A)) = A - cl(clA - A)$.

Finally, form the above remarks we have the following conditions corresponding to the Grzegorczyk logic (some of the points were announced in [6] without proof).

Theorem 2. *For every topological space X the following conditions are equivalent:*

i) $Grz \in E(X)$

ii) $A \subseteq cl(A - cl(clA - A))$, *for every* $A \subseteq X$,

iii) *If* $A \neq 0$, *then* $A - cl(clA - A) \neq 0$, *for every* $A \subseteq X$,

iv) *If* $A \neq 0$, *then* $A - \text{res}\, A \neq 0$, *for every* $A \subseteq X$,

v) *Every nonempty subset of the space X is resolvable into an alternated series of decreasing closed sets.*

A Boolean algebra is called *superatomic* (cf. [14]), if it is an atomic Boolean algebra and every its homomorphic image is atomic. Under Stone duality superatomic Boolean algebras correspond to scattered spaces in the sense that a Boolean algebra is superatomic iff its Stone space is scattered. A space is *scattered* iff it does not contain a dense-in-itself subset.

A topological space X is called a *Frechét* space if for every subset A of X, $a \in clA$ iff $a = \lim a_n$, for some sequence $\{a_n\}$ of points of A, (see [5]). A sequence $\{a_n : n \in \mathbb{N}\}$ will be denoted shortly by $\{a_n\}$.

A topological space X is called a *Urysohn* space, [5], if for every pair of distinct points x, y there exist open sets U i V such that $x \in U, y \in V$ and $clU \cap clV = \emptyset$.

Theorem 3. i) *For every Frechet space X, if X is a scattered space, then $Grz \in E(X)$.*

ii) *For every Frechet space X which is a Urysohn space: $Grz \in E(X)$ iff X is a scattered space.*

Proof. i) Assume, to the contrary, that $Grz \notin E(X)$. i.e. there is an $x \in A$ and $x \notin cl(A - cl(clAA))$, by Thm.2 ii). Hence, there is an open neighbourhood U of x such that $U \cap A - cl(clA - A) = \emptyset$, hence $x \in U \cap A \subseteq cl(clA - A)$. For every $a \in U \cap A$ there is a sequence $\{a_n\} \subseteq clA - A$ such that $a = \lim a_n$, $a \neq a_n$.

By suitable choice of a subsequence of the sequence $\{a_n\}$ one can check that the set $U \cap A$, is a dense-in-itself nonempty subset of X, a contradiction.

ii) Assume, to the contrary that X is not a scattered space, i.e. contains a dense-in-itself nonempty subset Y.

In order to prove that $Grz \notin E(X)$, that is (*) is false, it is enough to find in Y two disjoint sets, A and B and such that $A \subseteq clB$ and $B \subseteq clA$, for, in this case, $A \subseteq cl(clA - A)$, and hence $A - cl(clA - A) \subseteq A - A = \emptyset$, therefore Grz is false for any nonempty A.

In the dense-in-itself set Y we will construct a sequence $\{Z_n\}$ of sets in the following way:

Take $a \in Y$. Put $Z_0 = \{a\}$. (Stage 0). There is a sequence $\{b_n\} \subseteq Y$, $b_n \in a$ such that $a = \lim b_n$. By Urysohn condition subsequences of the sequence $\{b_n\}$ can be consecutively choosen in such a way that there is a subsequence $\{c_n\}$ of the sequence $\{b_n\}$ and a sequence of open neighbourhoods U_n of elements c_n such that U_n are pairwise disjoint. Put $Z_1 = \{c_n\}$. (Stage 1). (A sequence $\{d_{n,m} : m \in \mathbb{N}\}$ will be denoted shortly by $\{d_{n,m}\}$). This can be repeated: in every U_n there is a sequence $\{d_{n,m}\}$ such that $c_n = \lim d_{n,m}$, $d_{n,m} \neq c_n$ and there is a sequence of neighbourhoods $V_{n,m}$ of points $d_{n,m}$, pairwise disjoint. Put $Z_2 = \{d_{1,m}\} \cup \{d_{2,m}\} \cup \{d_{3,m}\} \cup ..$ (Stage 2). Again, in every neighbourhood $V_{n,m}$ there is a sequence $\{d_{n,m,k}\}$ such that $d_{n,m} = \lim d_{n,m,k}$, $d_{n,m} \neq d_{n,m,k}$ and there is a sequence of neighbourhoods $V_{n,m,k}$ of points $d_{n,m,k}$, pairwise disjoint. Put $Z_3 = \{d_{1,1,k}\} \cup \{d_{1,2,k}\} \cup \{d_{1,3,k}\} \cup \ldots \{d_{2,1,k}\} \cup \{d_{2,2,k}\} \cup \ldots \{d_{n,1,k}\} \cup \{d_{n,2,k}\} \cup \{d_{n,3,k}\} \cup \ldots \cup \ldots$ (Stage 3). This can be continued, and as a result we obtain the sets Z_n, Z_{n+1} such that $Z_n \cap Z_{n+1} = \emptyset$, $Z_n \subseteq clZ_{n+1}$, $n \in \mathbb{N}$. Now put: $A = Z_0 \cup Z_2 \cup Z_4 \cup \ldots$ and $B = Z_1 \cup Z_3 \cup Z_5 \cup \ldots$. The sets A and B have the required properties. q.e.d.

Corollary 2. *For every metric space X, $Grz \in E(X)$ iff X is a scattered space. In particular, for every countable Boolean algebra $\mathbf{A}$, $Grz \in E(\mathbf{St\ A})$ iff $\mathbf{A}$ is a superatomic Boolean algebra.*

Remark. Thm. 3 is not true, in general, if the space is not a Frechet one. There exist spaces X such that $Grz \in E(X)$ and X is a dense-in-itself space. For example so called SI - spaces, introduced by E.Hewitt* and constructed by means of Zorn's lemma, are of this kind. They do not contain two dense disjoint subsets (they are not compact, not Frechet, not metric).

C. Boolean algebras n-th derivative of which is a finite algebra.

Cantor-Bendixson derivative X^d of a topological space X is defined as the set of all points of condensation of X. By induction on ordinals Cantor-Bendixson derivative of rank α, for $\alpha \in On$, can be defined in a standard way:

$$X^{(0)} = X, \qquad X^{(\alpha+1)} = (X^{(\alpha)})^d, \qquad X^{(\lambda)} = \bigcap\{X^{(\alpha)} : \alpha < \lambda\},$$

for a limit ordinal λ.

By the Stone duality, Cantor-Bendixson derivative $\mathbf{A}^{(\alpha)}$ of rank α, for $\alpha \in On$ of a Boolean algebra $\mathbf{A}$, can be defined (see [14]).

For every Boolean algebra $\mathbf{A}$ let $I_{AT}(\mathbf{A})$ be the ideal of $\mathbf{A}$ generated by the atoms of $\mathbf{A}$. We define by induction, for any ordinal, an ideal I_α of $\mathbf{A}$. If I_α has been defined, put $\mathbf{A}^{(\alpha)} = \mathbf{A} / \mathbf{I}_\alpha$ the α–th *Cantor-Bendixson derivative* $\mathbf{A}^{(\alpha)}$ of A, and let $\pi_\alpha : \mathbf{A} \to \mathbf{A}^{(\alpha)}$ be the canonical homomorphism.

*E.Hewitt - A problem in set-theoretic topology, Duke Math. J. 10 (1943), 309-333.

Define: $\mathbf{I}_0 = \{0\}$, $\mathbf{I}_{\alpha+1} = \pi_\alpha^{-1}[\mathbf{I}_{AT}(\mathbf{A}^{(\alpha)})]$ and $\mathbf{I}_\lambda = \bigcup\{\, \mathbf{I}_\alpha : \alpha < \lambda\}$, for a limit ordinal.

Since $\mathbf{I}_\alpha \subset \mathbf{I}_{\alpha+1}$, it follows by induction that $\{\mathbf{I}_\alpha : \alpha \in On\}$ forms an increasing sequence of ideals of $\mathbf{A}$. Hence there is an ordinal α such that $\mathbf{I}_\alpha = \mathbf{I}_{\alpha+1}$.

Remark. Vanishing of n–th derivative of the Stone space, $\mathbf{St}\ \mathbf{A}^{(n)} = \emptyset$, is equivalent to the fact that $\mathbf{A}^{(n)}$ is a trivial Boolean algebra (consisting of one element). Moreover: $\mathbf{A}^{(n+1)}$ is trivial and $\mathbf{A}^{(n)}$ is non-trivial iff $\mathbf{St}\ \mathbf{A}^{(n+1)} = \emptyset$ and $\mathbf{St}\ \mathbf{A}^{(n)} \neq \emptyset$ iff $\mathbf{St}\ \mathbf{A}^{(n)}$ is a finite space iff $\mathbf{A}^{(n)}$ is a finite non-trivial algebra.

In what follows we say that $\mathbf{A}$ is a finite Boolean algebra iff $\mathbf{A}$ is a finite non-trivial Boolean algebra.

Let

$$B_0 = (p_0 \wedge \neg p_0), \qquad \text{i.e.} \qquad B_0 = \text{Falsum}$$

$$B_{n+1} = \Box p_{n+1} \vee \Box(\Box p_{n+1} \to B_n), \qquad n \in \mathbb{N}.$$

Define a sequence $\mathbf{Grz} + \mathbf{B}_n$, $n \in \mathbb{N}$, of normal modal logics which are extensions of Grzegorczyk logic $\mathbf{Grz}$ with the formulae B_n.

Lemma 1. *For every topological space X and valuation ν of variables in the TBA 2^X:*

a) $\nu(B_n) \supseteq X \setminus X^{(n)}$, *for* $n \geq 1$.

b) *For arbitrary scattered space X there is a valuation ν_0 such that* $\nu_0(B_n) = X \setminus X^{(n)}$, *for* $n \geq 1$.

Proof. a) By induction on n we prove: basic step 1^0) Let $a \in X \setminus \nu(B_1) = X \setminus \nu(\Diamond\Box p_1 \to \Box p_1) = cl[int\ \nu(p_1)] \setminus int\ \nu(p_1) = cl[(X \setminus \{a\}) \cap int\ \nu(p_1)] \setminus int\ \nu(p_1) \subseteq cl(X \setminus \{a\})$, i.e. $a \in X^{(1)}$.

Induction step 2^0) is based on an observation: $cl(intZ \cap X^{(n)}) \setminus intZ \subseteq X^{(n+1)}$, for every subset Z of X. We have $X \setminus \nu(B_{n+1}) = [X \setminus int\ \nu(p_{n+1})] \cap cl[int\ \nu(p_{n+1}) \cap (X \setminus \nu(B_n))] \subseteq cl[int\ \nu(p_{n+1}) \cap X^{(n)}] \setminus int\ (p_{n+1}) \subseteq X^{(n+1)}$.

b) In a scattered space $X int(A^\alpha) \neq \emptyset$; let $\nu_0(p_k) = X \setminus X^{(k)}$ for each k.

1^0. $X \setminus \nu_0(B_1) = clX^{(1)} \cap cl[X \setminus X^{(1)}] = X^{(1)} \cap X = X^{(1)}$. Similarly the induction step:

2^0. $X \setminus \nu_0(B_{n+1}) = clX^{(n+1)} \cap cl[X^{(n)} \setminus X^{(n+1)}] = X^{(n+1)}$. q.e.d.

Corollary 3. a) *If the n–th Cantor-Bendixson derivative of X vanishes, $X^{(n)} = \emptyset$, then $B_n \in E(X)$.*

b) *Moreover if X is a scattered space, then*

$$X^{(n)} = \emptyset \Leftrightarrow B_n \in E(X).$$

Corollary 4. a) *For every topological space X, if $X^{(n+1)} = \emptyset$, then* $\mathbf{Grz} + \mathbf{B}_{n+1} \subseteq E(X)$.

b) *if X is a scattered space, then*

$$X^{(n+1)} = \emptyset \text{ and } X^{(n)} \neq \emptyset \Leftrightarrow \mathbf{Grz} + \mathbf{B}_{n+1} \subseteq E(X)$$

and $B_n \notin E(X)$.

c) *For every countable Boolean algebra* $\mathbf{A}$:

$\mathbf{A}^{(n)}$ is a finite Boolean algebra $\Leftrightarrow \mathbf{Grz} + \mathbf{B}_{n+1} \subseteq E(StA)$

and $B_n \notin E(\mathbf{St\ A})$.

The proof follows immediately from Lemma 1 and Corollary 3.

Theorem 4. *For every countable Boolean algebra A the following conditions are equivalent:*

i) $\mathbf{A}^{(n)}$ *is a finite Boolean algebra*

ii) $E(\mathbf{St\ A}) = \mathbf{Grz} + \mathbf{B}_{n+1}$

Proof. First we present an indirect proof of Thm.4, as it gives a comparison with some other results in the area. After that we give a sketch of the more direct proof.

In view of Lem.2 b, it is enough to show that $E(\mathbf{St\ A}) \subseteq \mathbf{Grz} + \mathbf{B}_{n+1}$, if $\mathbf{A}^{(n)}$ is finite. The proof is based on some properties of the Tarski translation (McKinsey-Tarski [12]) between intuitionistic logic **INT** and modal logic **S4** and on the results of [4] on intermediate logics.

Tarski translation t from L_{INT} into L_M, where $(L_{INT}, \sim, \Rightarrow, \wedge, \vee)$ and $(L_M \neg, \rightarrow, \wedge, \vee, \Box)$ is the language of intuitionistic and modal logic, respectively, is defined as follows:

$t(p_i) = \Box t(p_i)$, for a propositional variable p_i,

$t(\sim \alpha) = \Box \neg t(\alpha)$,

$t(\alpha \Rightarrow \beta) = \Box(t(\alpha) \rightarrow t(\beta))$,

$t(\alpha \wedge \beta) = (t(\alpha) \wedge t(\beta))$

$t(\alpha \vee \beta) = (t(\alpha) \vee t(\beta))$, for $\alpha, \beta \in L_{INT}$.

McKinsey-Tarski [12] showed that: $\varphi \in \mathbf{INT} \Leftrightarrow t(\varphi) \in \mathbf{S4}$, for $\alpha \in L_{INT}$.

M.Dummett i E.J.Lemmon [2] (cf. also [16]) extended this result:

$\varphi \in \mathbf{INT} + \{\varphi_i : i \in I\} \Leftrightarrow t(\varphi) \in \mathbf{S4} + \{t(\varphi_i) : i \in I\}$, A.Grzegorczyk [7] found a proper extension **Grz** of **S4** logic such that: $\varphi \in \mathbf{INT} \Leftrightarrow t(\varphi) \in$ **Grz**. L.Maksimova i V. Rybakov [10] investigated modal companions of an intermediate logic , that is such modal **M** that, for $\alpha \in L_{INT} : \varphi \in \mathbf{L} \Leftrightarrow t(\varphi) \in \mathbf{M}$.

They showed that a set of modal companions of an intermediate logic L is infinite and contains the smallest logic $\tau\, \mathbf{L} = t(\mathbf{L})$ and the largest logic, denoted by $\sigma\, \mathbf{L}$. W.J. Blok [1], proved that: $\sigma\, \mathbf{L} = \tau\, \mathbf{L} + Grz$.

In [4], (cf. also [3]) a sequence of intermediate logics: $\mathbf{P}_n = \mathbf{INT} + P_n$ is considered, i.e. $\mathbf{P}_n$ is an extension of the intuitionistic logic **INT** with the formula P_n, and closed w.r.t. Modus Ponens and Substitution rules, where P_n is defined inductively:

$P_0 = (p_0 \wedge \sim p_0)$

$P_{n+1} = p_{n+1} \vee (p_{n+1} \Rightarrow P_n)$; in particular $P_1 = p_1 \vee \sim p_1$ (P_1 gives the classical logic) . It is shown (cf. Lemma 4 in [4]) that: for every metric and compact space X,

(**)

if $X^{(n)} \neq \emptyset$ and $X^{(n+1)} = \emptyset$ then $E(O(X)) = \mathbf{P}_{n+1}$, for $n \geq 0$,

where $O(X)$ is a Heyting algebra of open sets of the space X.

It is known (cf. [16], where $O(X)$ is denoted by $G(X)$) that for any space X, $\varphi \in E(O(X)) \Leftrightarrow t(\varphi) \in E(X)$, that is, $E(X)$ is a modal companion of $E(O(X))$. Hence $\mathbf{S4} + \mathbf{B}_{n+1} = \tau \mathbf{P}_{n+1} \subseteq E(X)$.

Now let $X = \mathbf{St\ A}$.

By means of the mentioned results of Maksimova - Rybakov and W.J. Blok we have:

$$\mathbf{S4} + \mathbf{B}_{n+1} \subseteq E(X) \subseteq \mathbf{Grz} + \mathbf{B}_{n+1}$$

and in view of Corollary 2: $\mathbf{Grz} \subseteq E(X)$; finally we have $E(X) = \mathbf{Grz} + \mathbf{B}_{n+1}$. q.e.d.

For more direct proof Kripke strucures are used. In particular we use the completeness result for the logic **Grz** with respect to finite trees (finite partial orders) which is due to K. Segerberg [17] . We also apply the following property of any Kripke structure F, that verifies the formula B_n:

$(***)$ $\quad B_n \in E(F)$ iff height of F is not greater then n, where F is a Kripke structure with the relation R, $E(F)$ denotes the set of all formulae valid in F. Height of F is defined as the number of elements in a maximal R-chain $a_1, \ldots, a_n$ in F (i.e. $a_i R a_{i+1}$ and not $a_{i+1} R a_i$, for all $i < n$). This property can be proved by induction on n.

Let $\varphi \notin \mathbf{Grz} + \mathbf{B}_{n+1}$. Form the Segerberg completeness theorem for **Grz** there exists a finite Kripke structure , which is a tree and which verifies **Grz** and B_{n+1} and such that $\varphi \notin E(F)$. By $(***)$ the height of F is not greater then $n+1$.

We prove that $\varphi \notin E(\mathbf{St\ A})$ in two steps.

1^0. We use Kripke structures $R_{n+1,m}$ which are trees of the height equal to $n+1$ and which in every node, excluding the final node, have exactly m immediate successors (cf. eg. Ono [15]). By the similar reasoning as in the proof of Lem.4 in [4] we show that for every $m \in \mathbb{N}$ there is an open and continuous mapping (so called interior mapping, Ch.III, par.3 in [16]) of the space $\mathbf{St\ A}$ (for which $\mathbf{St\ A}^{(n+1)} = \emptyset$ and $\mathbf{St\ A}^{(n)} \neq \emptyset$) onto the tree $R_{n+1,m}$. Hence $E(\mathbf{St\ A}) \subseteq E(R_{n+1,m})$.

Remark. Validity in $R_{n+1,m}$ as a Kripke structure coincide with the validity in a topological space $R_{n+1,m}$ with the topology of R-closed (upward) sets.

2^0. Take a tree F of the height not greater then $n+1$ and corresponding tree $R_{n+1,m}$. It can be defined a p–morphism from $R_{n+1,m}$ onto F. Hence $E(R_{n+1,m}) \subseteq E(F)$. Finally, from 1^0 and 2^0, $\varphi \notin E(\mathbf{St\ A})$. q.e.d.

Corollary 5. *For* $\mathbf{Tr} = \mathbf{Grz} + \mathbf{B}_1 = \mathbf{Grz} + (\Diamond\Box p_1 \to \Box p_1) = \mathbf{Grz} + \mathbf{S5}$ *we have* $E(\mathbf{St\ A}) = \mathbf{Tr}$ *iff* $\mathbf{A}$ *is a finite Boolean algebra.*

Example 1. *If* $\mathbf{A}$ *is the algebra of finite - cofinite subsets of the set of natural numbers* $\mathbb{N}$, *then* $\mathbf{St\ A}$ *is the one point compactification of the set* $\mathbb{N}$, *(Aleksandrov compactification) which is the least compactification of* $\mathbb{N}$. *Then*

$$E(\mathbf{St\ A}) = \mathbf{Grz}+\mathbf{B}_2 = \mathbf{Grz} + \Box p_2 \vee \Box(\Box p_2 \to (\Diamond\Box p_1 \to \Box p_1)).$$

Example 2. $E((\omega^n + 1)) = \mathbf{Grz} + \mathbf{B}_{n+1}$, *for* $n \geq 0$, *where* $(\omega^n + 1)$ *is the topological space on the ordinal* $\omega^n + 1$ *with the order topology.*

Proof. In the proof of Thm.4 we use Corollary 2 of [4]: $\mathbf{P}_{n+1} = E(O(\omega^n + 1))$, $n \geq 0$; hence $\mathbf{S4} + \mathbf{B}_{n+1} = \tau\ \mathbf{P}_{n+1} \subseteq E(\omega^n + 1)$. By means of the mentioned results of Maksimova - Rybakov and W. J.Blok we have: $\mathbf{S4} + \mathbf{B}_{n+1} \subseteq E(\omega^n + 1) \subseteq \mathbf{Grz} + \mathbf{B}_{n+1}$, and by Corollary 1: $\mathbf{Grz} \subseteq E(\omega^n + 1)$, hence $E(\omega^n + 1) = \mathbf{Grz} + \mathbf{B}_{n+1}$. This can be also proved directly, similarly to the proof of Thm.4. q.e.d.

Corollary 6. *For every compact and scattered metric space* X, *if* $X^{(n)} \neq \emptyset$, *for every* $n \in \mathbb{N}$, *(in particular if* $X^{(\alpha)} \neq \emptyset$ *and* $\alpha \geq \omega$*), then* $E(X) = \mathbf{Grz}$.

Proof. From the theorem of Mazurkiewicz and Sierpiński [13] the space X is homeomorphic to the space $(\omega^\alpha \bullet k + 1)$ for some countable ordinal α and $n \in \mathbb{N}$. (Remark: $\omega^0 = 1$). Since for every $n \in \mathbb{N}$ the space $(\omega^n + 1)$ is an open subset of $(\omega^\alpha \bullet k + 1)$, hence (compare Example 2) we have:

$$E(X) = E(\omega^\alpha \bullet k + 1) \subseteq$$
$$\subseteq E(1+1) \cap E(\omega + 1) \cap E(\omega^2 + 1) \cap E(\omega^3 + 1) \cap \ldots =$$
$$= \mathbf{Grz} + \mathbf{B}_1 \cap \mathbf{Grz} + \mathbf{B}_2 \cap \mathbf{Grz} + \mathbf{B}_3 \cap \ldots = \mathbf{Grz}$$

We prove the last equality. Assume that $\varphi \notin \mathbf{Grz}$. Form the Segerberg completeness theorem for **Grz** there exists a finite Kripke structure F, which is a tree and which verifies **Grz** and such that $\varphi \notin E(F)$. Since the structure F is finite, it has a finite hight, say $k \in \mathbb{N}$. From $(***)$ in the proof of Thm.4 we have $\mathbf{Grz} + \mathbf{B}_k \subseteq E(F)$. Consequently $\varphi \notin \mathbf{Grz} + \mathbf{B}_k$ for some k and this completes the proof.

Corollary 7. *For every countable and superatomic Boolean algebra* $\mathbf{A}$, *if* $\mathbf{A}^{(n)}$ *is infinite, for every* $n \in \mathbb{N}$, *(in particular, if* $\mathbf{A}^{(\alpha)}$ *is an infinite algebra for some* $\alpha \geq \omega$*), then* $E(\mathbf{St\ A}) = \mathbf{Grz}$.

Finally we have:

Corollary 8. *The chain of the type* $\omega + 1$ *of modal logics:*

$$\mathbf{Tr} = \mathbf{Grz} + \mathbf{B}_1 \supset \mathbf{Grz} + \mathbf{B}_2 \supset \mathbf{Grz} + \mathbf{B}_3 \supset \ldots \supset \mathbf{Grz}$$

contains all and only such modal logics that can be obtained as $E(\mathbf{St\ A})$ *for an arbitrary countable and superatomic Boolean algebra* $\mathbf{A}$.

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