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The sum operation and link lattices

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Abstract

The sum operation, as introduced by Andrzej Wroński, is used to decompose any finite distributive lattice into its Boolean fragments. The decomposition is not unique but its maximal components are uniquely determined. We define an ordering relation between these maximal Boolean fragments of a given lattice and use this link ordering to describe the structure of the lattice.

Suppose that $\mathcal{A} = <A, \leq_A>$ and $\mathcal{B} = <B, \leq_B>$ are lattices such that $A \cap B$ is a filter in $\mathcal{A}$ and an ideal in $\mathcal{B}$. Assume that the orderings $\leq_A$ and $\leq_B$ coincide on $A \cap B$ and let us define a binary relation $\leq$ on $A \cup B$ by

$$\leq_A \cup \leq_B \cup (\leq_A \circ \leq_B)$$

Theorem 1 *The relation $\leq$ is a lattice ordering on $A \cup B$ such that*

(1) $a \vee b = a \vee_A b$ *and* $a \wedge b = a \wedge_A b$ *if* $a, b \in A$;

(2) $a \vee b = a \vee_B b$ *and* $a \wedge b = a \wedge_B b$ *if* $a, b \in B$;

(3) $a \vee b = (a \vee_A c) \vee_B b$ *if* $a \in A \setminus B$, $b \in B \setminus A$ *and* $c \in A \cap B$ *is any element such that* $c \leq_B b$

(4) $a \wedge b = (b \wedge_B c) \wedge_A a$ *if* $a \in A \setminus B$, $b \in B \setminus A$ *and* $c \in A \cap B$ *is any element such that* $c \geq_A a$

Proof. One easily shows that the just defined relation is an ordering on $A \cup B$ which coincides with $\leq_A$ on A and and it coincides with $\leq_B$ on B. In result, (1) and (2) clearly hold.

To get (3) one needs to show that $(a \vee_A c) \vee_B b$ does not depend on the choice of the element c. So, let us prove

(5) $(a \vee_A c) \vee_B b = (a \vee_A c') \vee_B b$ if $a \in A \setminus B$, $b \in B \setminus A$ and $c, c' \in A \cap B$ are any elements such that $c, c' \leq_B b$

Without loosing of generality we may assume that $c \leq_B c'$. If this were not fulfilled one might introduce a third element $c'' = c \wedge_B c'$ for which $c \leq_B c''$, $c' \leq_B c''$ and $c'' \leq_B b$. Then one would get

$$(a \vee_A c) \vee_B b = (a \vee_A c'') \vee_B b = (a \vee_A c') \vee_B b.$$

Since the orderings $(\leq_A)$ and $(\leq_B)$ coincide on $A \cap B$, we also get $c \leq_A c'$. Then by monotonicity of the lattice operations $a \vee_A c \leq_A a \vee_A c'$. As $A \cap B$ is a filter in $\mathcal{A}$, then $a \vee_A c, a \vee_A c' \in A \cap B$. Thus $a \vee_A c \leq_B a \vee_A c'$ and, by monotonicity of $\vee_B$, it follows that

$$(a \vee_A c) \vee_B b \leq_B (a \vee_A c') \vee_B b.$$

To show the reverse, let us introduce a fresh element

$$d = (a \vee_A c') \wedge_B ((a \vee_A c) \vee_B b)$$

Since $c' \leq_A a \vee_A c'$ and $c' \in A \cap B$, then $c' \leq_B a \vee_A c'$. Next, $c' \leq_B b \leq_B (a \vee_A c) \vee_B b$ and hence by the definition of the meet operation in the lattice, $c' \leq_B d$. We get $d \in A \cap B$ as $d \leq_B a \vee_A c' \in A \cap B$ and it means that $c' \leq_A d$. Let us note, moreover, that $a \vee_A c \leq_B (a \vee_A c) \vee_B b$. Hence $a \vee_A c \leq_B d$ by the definition of the element d. Then

$$a \leq_A a \vee_A c \leq_A d$$

and by the definition of the join operation in $\mathcal{A}$, one gets $a \vee_A c' \leq_A d$. This yields $a \vee_A c' \leq_B d$ and hence $a \vee_A c' \leq_B (a \vee_A c) \vee_B b$ which suffices to get

$$(a \vee_A c') \vee_B b \leq_B (a \vee_A c) \vee_B b.$$

The proof of (5) is completed. Our further argument for (3) is standard. Clearly, $(a \vee_A c) \vee_B b$ is a bound of a and b in $A \cup B$. Let x be any other bound. As $b \leq x$, we get $b \leq_B x$ and $x \in B \setminus A$. By $a \leq x$, there is an element $y \in A \cap B$ such that $a \leq_A y$ and $y \leq_B x$. Take $c = b \wedge_B y$. Then $c \leq_B b$ and $c \in A \cap B$. As $c \leq_A y$, by the definition of the join operation $a \vee_A c \leq_A y$ which yields $a \vee_A c \leq_B y$ and hence $a \vee_A c \leq_B x$. In addition, $b \leq_B x$ and hence $(a \vee_A c) \vee_B b \leq_B x$, which completes our argument.

Eventually, let us note that (4) follows from (3) by lattice duality □

The resulting lattice, called a sum of $\mathcal{A}$ and $\mathcal{B}$, is denoted by $\mathcal{A} \oplus \mathcal{B}$. The sum operation was introduced by Wroński [7], and its special case with $A \cap B = \{\mathbf{1}_A\} = \{\mathbf{0}_B\}$ by Troelstra [6]. In particular, if $\mathcal{B}$ is a two–element Boolean algebra then $\mathcal{A} \oplus \mathcal{B}$ is the same as $\mathcal{A}\oplus$, where $\oplus$ is the Jaśkowski operation of adding to $\mathcal{A}$ the top element (so called "mast"), see [4].

Kotas, Wojtylak [3] proved that the closure of the class of all finite Boolean algebras with respect to the sum operation is the class of all finite distributive lattices. It means that for every finite distributive lattice $\mathcal{D}$ there is a finite family $\{\mathcal{B}_i\}_{i \in T}$ of Boolean algebras such that $\mathcal{D}$ is a sum of that family. In this case we shall write $\mathcal{D} = \oplus\{\mathcal{B}_i\}_{i \in T}$. As the sum operation is nonassociative and noncommutative, this notation does not give us any clue about the ordering in which the summation should be performed. There is no uniqueness of doing it. Despite of this, we are going to prove that all maximal elements of the decomposition are uniquely determined.

Let $\mathcal{D}$ be a lattice. A sublattice $\mathcal{D}_1$ of $\mathcal{D}$ is said to be a fragment of $\mathcal{D}$, and we shall write $\mathcal{D}_1 \sqsubseteq \mathcal{D}$ in this case, if

$$a \leq c \leq b \quad \text{and} \quad a, b \in D_1 \Rightarrow c \in D_1, \qquad \text{for every } a, b, c \in D.$$

If, additionally, $\mathcal{D}_1$ is a Boolean lattice, then $\mathcal{D}_1$ is said to be a Boolean fragment of $\mathcal{D}$. Any filter (ideal) of $\mathcal{D}$ is its fragment and, if $\mathcal{D}$ is finite, then any fragment of $\mathcal{D}$ is its interval, that is a set of the form $\{x \in D : a \leq x \leq b\}$ for some a and b. Clearly, $\sqsubseteq$ is transitive and $D_i \sqsubseteq D_1 \oplus D_2$ for $i = 1, 2$.

Theorem 2 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be lattices and $\mathcal{B}$ be a complementary lattice. If $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$ then $\mathcal{B} \sqsubseteq \mathcal{D}_1$ or $\mathcal{B} \sqsubseteq \mathcal{D}_2$.*

Proof. Let $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$ and assume that there are elements $a \in B \setminus D_2$ and $b \in B \setminus D_1$. Let $\mathbf{0}_B$ be the zero of $\mathcal{B}$ and $\mathbf{1}_B$ be the unit of $\mathcal{B}$. Since $\mathbf{0}_B \leq a$ and $a \in D_1$, then $\mathbf{0}_B \in D_1 \setminus D_2$. Similarly, we conclude that $\mathbf{1}_B \in D_2 \setminus D_1$. Since $\mathbf{0}_B \leq \mathbf{1}_B$, there is an element $x \in D_1 \cap D_2 \cap B$ by the definition of the ordering in $\mathcal{D}_1 \oplus \mathcal{D}_2$. Let $y \in B$ be the complement of x in $\mathcal{B}$. If $y \in D_1$ then $\mathbf{1}_B = x \vee y \in D_1$, which is impossible. Otherwise $y \in D_2$, so $\mathbf{0}_B = x \wedge y \in D_2$, which also contradicts our assumptions. Thus, $\mathcal{B} \subseteq \mathcal{D}_1$ or $\mathcal{B} \subseteq \mathcal{D}_2$, and it yields $\mathcal{B} \sqsubseteq \mathcal{D}_1$ or $\mathcal{B} \sqsubseteq \mathcal{D}_2$ as $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$. □

Since every finite distributive lattice $\mathcal{D}$ is a sum $\oplus$ of Boolean lattices $\mathcal{B}_i$, then D is also the set-theoretical sum $\bigcup B_i$ of its Boolean fragments. The components of the sum are not uniquely determined as it is possible that $B_i \subseteq B_j$ for some i, j. We call a family $\{\mathcal{B}_i\}$ of Boolean lattices a scarce decomposition of $\mathcal{D}$ iff $\mathcal{D} = \bigoplus \mathcal{B}_i$ and $\mathcal{B}_i \subseteq \mathcal{B}_j$ does not hold for any $i \neq j$. As the immediate Corollaries of Theorem 2 we get:

Corollary 1 *If a lattice $\mathcal{D}$ is a sum $\oplus$ of a family $\{\mathcal{B}_i\}$ of Boolean lattices, then the family contains all maximal Boolean fragments of $\mathcal{D}$.*

Corollary 2 *There is at most one scarce decomposition of any finite distributive lattice $\mathcal{D}$ and the decomposition consists of all maximal Boolean fragments of $\mathcal{D}$.*

It may happen, however, that a finite distributive lattice $\mathcal{D}$ does not have the scarce decomposition. More specifically, it is sometimes impossible to get $\mathcal{D}$ as a sum $\bigoplus$ of its maximal Boolean fragments without taking subalgebras of maximal fragments or repeating them in the sum operations.

Let us consider $K = \{\mathcal{B}_i\}_{i \in I}$ the family of all maximal Boolean fragments of a finite distributive lattice $\mathcal{D} = < D, \leq >$ and the ordering relation on K defined in the following way:

$$\mathcal{B} \preceq \mathcal{B}' \text{ iff } \mathbf{1}_B \leq \mathbf{1}_{B'}$$

for any $\mathcal{B}, \mathcal{B}'$ from K.

It can be proved (see [2]) that

Theorem 3 *$\mathcal{K} = < K, \preceq >$ is a lattice, where, for any $\mathcal{B}_1, \mathcal{B}_2$ from K, the infimum $\mathcal{B}_1 \wedge \mathcal{B}_2$ is determined by all $(\mathbf{1}_1 \wedge \mathbf{1}_2)$-atoms and the supremum $\mathcal{B}_1 \vee \mathcal{B}_2$ is determined by all $(\mathbf{0}_1 \vee \mathbf{0}_2)$-atoms. $\mathbf{1}_1, \mathbf{0}_1, \mathbf{1}_2, \mathbf{0}_2$ denote, respectively, the units and zeroes of the maximal Boolean fragments $\mathcal{B}_1$ and $\mathcal{B}_2$.*

We shall call the lattice $\mathcal{K}$ the link lattice of $\mathcal{D}$.

Figure 1 shows an example of a distributive lattice with its link lattice.

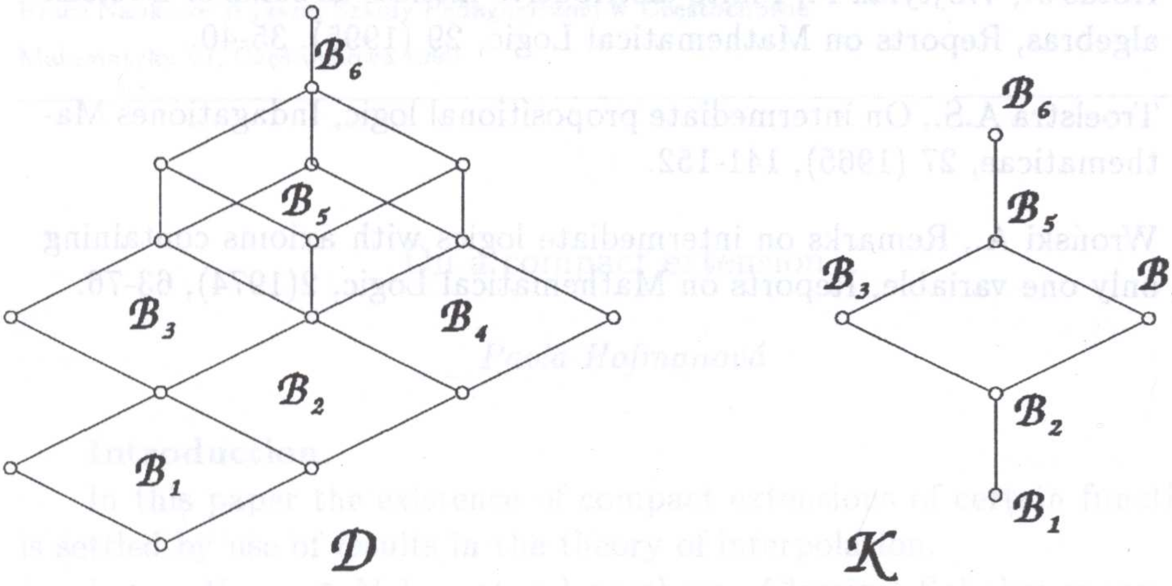

Figure 1

The link lattices can be applied to solving the problem of scarce decomposition, because, as we proved in [2]:

Theorem 4 *A finite distributive lattice $\mathcal{D}$ has a scarce decomposition iff every (at least two-element) fragment of its link lattice $\mathcal{K}$ contains a prime ideal.*

It is easy to observe that the lattice $\mathcal{D}$ in Figure 1 has a scarce decomposition.

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