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On a compact extension

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Introduction.

In this paper the existence of compact extensions of certain functions is settled by use of results in the theory of interpolation.

Let $n, N, p, q \in \mathbf{N}$ be natural numbers. Classical Sobolev spaces are denoted by the letter W with or without indices.

Theorem 1 *Let $p > 1, \Omega \in C^{0,1}$. Then there exists a continuous linear mapping T from $W^{1-1/p,p}(\partial\Omega)$ into $W^{1,p}(\Omega)$ such that for $v = Tu$ if $v = u$ in $\partial\Omega$ (see [2], page 341, 6.9.2 theorem).*

Theorem 2 *Let $0 < \Theta < 1$. Moreover, let $s, s_0, s_1, p, p_0, p_1, q, q_0, q_1$ and r be numbers such that*

$$\begin{aligned} s^* &= (1-\Theta)s_0 + \Theta s_1, \\ \frac{1}{p^*} &= \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \\ \frac{1}{q^*} &= \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}. \end{aligned}$$

Then we have

$$(H_p^{s_0}, H_p^{s_1})_{\Theta,q} = B_{pq}^{s^*}, \text{ for } s_0 \neq s_1, 1 \leq p, q \leq \infty.$$

(see [1], page 152, 6.4.5 theorem.)

Theorem 3 (The embedding theorem) *Assume that $s - n/p = s_1 - n/p_1$. Then the following inclusions hold*

$$\begin{aligned} B_{pq}^s &\subset B_{p_1 q_1}^{s_1} \quad (1 \leq p \leq p_1 \leq \infty, 1 \leq q \leq q_1 \leq \infty, s, s_1 \in \mathbf{R}), \\ H_p^s &\subset H_{p_1}^{s_1} \quad (1 \leq p \leq p_1 \leq \infty, s, s_1 \in \mathbf{R}). \end{aligned}$$

(see [1], page 153, 6.5.1 theorem.)

Theorem 4 *Let B be a Banach space and (A_0, A_1) a couple of Banach spaces. Let T be a linear operator.*

(i) Assume that

$$T : A_0 \to B \text{ compactly},$$
$$T : A_1 \to B,$$

and that E belongs to the class $C_K(\Theta; \overline{A})$ for some Θ with $0 < \Theta < 1$. Then

$$T : B \to E \text{ compactly}.$$

(ii) Assume that

$$T : B \to A_0 \text{ compactly},$$
$$T : B \to A_1,$$

and that E belongs to the class $C_J(\Theta; \overline{A})$ for some Θ with $0 < \Theta < 1$. Then

$$T : E \to B \text{ compactly}.$$

(see [1], page 56, 3.8.1 theorem.)

These theorems are used in proofs of the following two theorems.

Theorem 5 *Let $\Gamma \in R^{n-1}$. Then the embedding of the space $W^{1,p}(\Gamma)$ into the space $W^{1-1/q,q}(\Gamma)$ is compact for $n - 1 < p \leq q < \dfrac{np}{n-1}$.*

Proof: First of all we will use theorem 4. The embedding $W^{1,p} \to W^{1,p}$ is continuous and the embedding $W^{1,p} \to L^p$ is compact. We obtain the continuous embedding $W^{1,p} \to B^s_{pq}$ where B^s_{pq} is a Besov space with $0 < s < 1$, $q > 1$. Further we use the theorem 3 and so we have the compact embedding of the Besov space B^s_{pq} into the Besov space B^s_{pq}. Now we have the compact embedding

$$W^{1,p} \to B^s_{pq}.$$

We will use theorem 2 and we will get the embedding $B^s_{pq} \to B^{1-1/q}_{qq}$ for

$$s = 1 - \frac{1}{q} - \frac{n-1}{q} + \frac{n-1}{p}.$$

Since $0 < s < 1$, this fact follows directly from our premisses. The following relations hold if k is not whole:

$$W^{k,q} = B^k_{qq}.$$

This fact we can find in a remark in [2], page 390. Now proof of the theorem 5 is completed.

Theorem 6 (The compact extension) *Let be $\Omega \in C^{0,1}, \Omega \subset \mathbf{R}^n$ and a number $q < \frac{np}{n-1}$. Then there exists the compact extension operator*

$$\tilde{T} : W^{1,p}(\partial\Omega, \mathbf{R}^N) \to W^{1,q}(\Omega, \mathbf{R}^N).$$

Proof: We can assume that $\Omega = \cup\Gamma_i$, $i = 1, \dots, l$, $l \in \mathbf{N}$ $\Gamma_i \subset \mathbf{R}^{n-1}$. Our proof is completed using theorems 5 for every $i = 1, \dots, l$ and theorem 1.

References

[1] Berg, Löfström, *Interpolation Spaces.*

[2] A. Kufner, O. John, S. Fučík, *Function Spaces,* Academia, Praha 1977.

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