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n–dimensional affine ratio

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Let X be the m–dimensional affine space ($m \in \mathbb{N}$) and let $n \in \mathbb{N}$, $1 \leq n \leq m$, $a_o, a_1, \ldots, a_n, p \in X$. Assume that

(i) the points $a_o, a_1, \ldots, a_n, p$ are different and belong to n–dimensional affine subspace,

(ii) $dim Af(a_1, \ldots, a_n) = n - 1$ and $p \notin Af(a_1, \ldots, a_n)$,
(where by $Af(a_1, \ldots, a_n)$ we denote the affine subspace generating by $a_1, \ldots, a_n$).

Definition 1 The n–dimensional simple ratio of points $a_o, a_1, \ldots, a_n$ by the point p is a finite sequence of numbers $(\alpha_1, \ldots, \alpha_n)$ such that

$$\overrightarrow{a_o p} = \alpha_1 \overrightarrow{a_1 p} + \ldots + \alpha_n \overrightarrow{a_n p}$$

and it is denoted by

$$s(a_o, a_1, \ldots, a_n; p) := (\alpha_1, \ldots, \alpha_n).$$

Remark 1 If $n = 1$, theb we obtain a definition of the cross - ratio

$$s(a_o, a_1; p) = \alpha_1 \Leftrightarrow \overrightarrow{a_o p} = \alpha_1 \overrightarrow{a_1 p}.$$

Remark 2 If $n = 2$, then we obtain a definition of the planar simple ratio

$$s(a_o, a_1, a_2; p) = (\alpha_1, \alpha_2) \Leftrightarrow \overrightarrow{a_o p} = \alpha_1 \overrightarrow{a_1 p} + \alpha_2 \overrightarrow{a_2 p}.$$

Remark 3 Moreover, if for every $i \in \{1, \ldots, n\}$

$$a_i \notin Af(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),$$

then $\alpha_i \neq 0$.

Let $I = \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$.

Definition 2 A finite set $\{a_i : i \in I\}$ of points in X is affinely independent, if

$$\bigwedge_{i_o \in I} a_{i_o} \notin Af(a_i)_{i \in I_o},$$

where $I_o = I \setminus \{i_o\}$.

Theorem 1 Let $\{a_o, a_1, \ldots, a_n\}$ be an affinely independent set of $(n+1)$ points in X. The following statements are equivalent:

1) a point p is a centre of gravity of n–simplex $\Delta(a_o, a_1, \ldots, a_n)$ with vertices $a_o, a_1, \ldots, a_n$,

2) $s(a_o, a_1, \ldots, a_n; p) = (-1, \ldots, -1)$.

Corollary 1 If $n = 1$, then a point p is the centre of the interval $\overline{a_o a_1}$ if and only if

$$s(a_o, a_1; p) = -1.$$

Corollary 2 If $n = 2$, then a point p is the centre of gravity of the triangle $\Delta a_o a_1 a_2$ if and only if

$$s(a_o, a_1, a_2; p) = (-1, -1).$$

Theorem 2. Let $\{a_o, a_1, \ldots, a_n\}$ be an affinely independent set of $(n+1)$ points in X.
The following statements are equivalent

1) $p \in \operatorname{int} \Delta(a_o, a_1, \ldots, a_n)$,

2) $\bigvee_{\alpha_1, \ldots, \alpha_n \in \mathbb{R}} (s(a_o, a_1, \ldots, a_n; p) = (\alpha_1, \ldots, \alpha_n) \wedge \alpha_i < 0,\ i = 1, \ldots, n)$

Corollary 1 If $n = 1$, then

$$p \in \operatorname{int} \overline{a_o a_1} \Leftrightarrow s(a_o, a_1; p) < 0.$$

Corollary 2 If $n = 2$, then

$$P \in \operatorname{int} \Delta a_o a_1 a_2 \Leftrightarrow \bigvee_{\alpha_1, \alpha_2 \in \mathbb{R}} Is(a_o, a_1; p) = (\alpha_1, \alpha_2) \wedge \alpha_1 < 0 \wedge \alpha_2 < 0.$$

Lemma 1 Let the points $a_o, a_1, \ldots, a_n, p$ fulfil (i), (ii), (iii) and

$$s(a_o, a_1, \ldots, a_n; p) = (\alpha_1, \ldots, \alpha_n).$$

Then

1) $$\bigwedge_{i\in\{1,\ldots,n\}} s(a_i, a_1, \ldots, a_{i-1}, a_o, a_{i+1}, \ldots, a_n; p) =$$
$$= \left(-\frac{\alpha_1}{\alpha_i}, \ldots, -\frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i}, -\frac{\alpha_{i+1}}{\alpha_i}, \ldots, -\frac{\alpha_n}{\alpha_i}\right)$$

2) if $\alpha_1 + \ldots + \alpha_n \neq 1$, then

$$s(p, a_1, \ldots, a_n; a_o) = \left(\frac{\alpha_1}{\sum\limits_{i=1}^{n} \alpha_{i-1}}, \ldots, \frac{\alpha_n}{\sum\limits_{i=1}^{n} \alpha_{i-1}}\right).$$

Corollary 1 If $n = 1$, then

$$s(a_1, a_o; p) = \frac{1}{s(a_o, a_1; p)}$$

Corollary 2 If $n - 2$, then

$$s(a_1, a_o, a_2; p) = \left(\frac{1}{\alpha_1}, -\frac{\alpha_2}{\alpha_1}\right)$$

and

$$s(a_2, a_1, a_o; p) = \left(-\frac{\alpha_1}{\alpha_2}, \frac{1}{\alpha_2}\right).$$

Lemma 2 Let the points $a_o, a_1, \ldots, a_n, p)n = 2)$ fulfil (i) and (ii). Then the following statements are equivalent

1) $a_o \in Af(a_1, \ldots, a_n)$

2) $$\bigvee_{\alpha_1,\ldots,\alpha_n\in\mathbb{R}} (s(a_o, a_1, \ldots, a_n; p) = (\alpha_1, \ldots, \alpha_n) \wedge \sum_{i=1}^{n} \alpha_i = 1).$$

Corollary 3 If $n = 2$, then the points a_o, a_1, a_2 are collinear if and only if there exist a number α such that

$$s(a_o, a_1, a_2; p) = (\alpha, 1 - \alpha).$$

References

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