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Some Undecidable Statements of Quite Simple Mathematics

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Abstract

There are well known problems which are not decidable: halting problem, provability in PA, being a first order tautology, and others. Since all these problem deal with notions like computability and provability, they are beyond the scope of ,,usual" mathematics – mathematical analysis, for instance. Here, we will show a bunch of examples of simple undecidable statements of such mathematics.

1. Introduction.

It is well known that:

Theorem 1.1 (A. Tarski, 1951, [3]) For every real-closed field $\mathcal{F}$ the theory $\mathsf{Th}(\mathcal{F})$ is decidable.

As an immediate corollary of the above we obtain:

Corollary 1.2 The problem of solvability of polonomial equations is decidable in every real closed field.

However, the same is not valid in rings. We have the following:

Theorem 1.3 (J. Matjasiewicz, 1970, [1]) Every recursively enumerable relation is diofantic, i.e. is of the form:

$$\{(n_1, \ldots, n_N) : \exists_{m_1,\ldots,m_M}(W(m_1, \ldots, m_M, n_1, \ldots, n_N) = 0)\},$$

for some polynomial W with integer coefficients.

Since there exists nonrecursive recursively enumerable relation, we get:

Corollary 1.4 The problem of solvability of polynomial equations is not decidable in the theory of integer numbers.

Because, (see [2]), being an integer is definable in the arithmetic of rationals, we also have:

Corollary 1.5 The problem of solvability of polynomial equations is not decidable in the theory of rationals.

The above also gives that:

Corollary 1.6 The properties of being an integer and of being a rational are not definable in arithmetics of reals.

There are also other undecidable problems which deal with real numbers and sequences of reals. We list some of them below and the proofs will be given in the next paragraph. First, let us accept the following definition:

Definition 1.7 Let $\delta : \omega \to \omega$ and let $b \in \omega\backslash 2$.
Then $[\![\delta]\!]_b \stackrel{\mathrm{DF}}{=} \sum_{n\in\omega}(\delta_n \bmod b)/b^{n+1}$. The number $[\![\delta]\!]_b$ will be called the value of the sequence δ with the basis b. Given a positive real r and $b \in \omega\backslash 2$, the sequence $\delta_b(r) : \omega \Rightarrow \omega$ is an b-adic expansion of r iff $r = [\![\delta_b(r)]\!]_b$.
A real $r \in <0, 1>$ is recursive iff $\delta_2(r)$ is a recursive function.

Of course, the choice of the basis 2 in the above is quite arbitrary — we easily notice that:

Remark 1.8 A real r is recursive iff $\delta_b(r)$ is a recursive function, for every $b \in \omega\backslash 2$.

Theorem 1.9 Let r be recursive. The following problems are undecidable:

- r is rational.
- r is algebraic of degree $n \in \omega\backslash 2$.
- r contains at least one [does not contain any] occurence of the digit 1 in its b-adic expansion, $b \in \omega\backslash 2$.
- r equals 0.

Definition 1.10 Let $\chi : \omega \to \mathbb{R}$ be a sequence of recursive reals. The sequence χ is recursive iff $\delta_b[\chi] : \omega^2 \to \omega$ is recursive, where $\delta_b[\chi](n, k) \stackrel{DF}{=} \delta_b(\chi_n)_k$, $n, k \in \omega$.

We have:

Theorem 1.11 Let χ be a sequence of recursive reals. The following problems are not decidable:

- χ is convergent.
- χ has no accumulation points.
- r is an accumulation point of χ.
- χ is monotone.
- χ is constant.

2. Proofs.

Let $F : \omega \to \omega$ be a fixed one-to-one recursive function with a non-recursive image (e.g. an enumeration of Gödel numbers of PA theorems).

Definition 2.1

Let α and β be any sequences and let, then,

$$H(\alpha,\beta,M)_k = \begin{cases} \alpha_k, & \text{iff } M \in F^{\shortparallel}k, \\ \beta_k, & \text{otherwise} \end{cases}$$

We see that $H(\alpha,\beta,M)$ is recursive, provided α and β are recursive sequences of integers or recursive reals. Moreover, the mapping transforming recursive α, β and a number M to the sequence $H(\alpha,\beta,M)$ is also recursive. I.e. there exists an algorithm which given finite descriptions of α, β and M makes a „program" which computes $H(\alpha,\beta,M)$.

Remark 2.2 If α and β are not almost everywhere equal, then

$$H(\alpha,\beta,M) = \beta \Leftrightarrow M \in F^{\shortparallel}\omega.$$

Definition 2.3 Let $\alpha,\beta : \omega \to \omega$ and let $\mathcal{W}$ be a decidable property of reals. Then let $\Pi_{\alpha,\beta,\mathcal{W}}$ be the following function-program (we use here a Pascal-like notation):

```
function Π_{α,β}(m : Nat):boolean;
var c: Nat;
begin
    c:=codeOf(H(α,β,M));
    Π_{α,β}:=is𝒲(c);
end.
```

where Nat denotes the type of positive integers.

Proof. (Of Theorem 1.9)

(•) (r is rational)

Let $\alpha \overset{\text{DF}}{=} \delta(\sqrt{2}-1)$ and let β be contstantly equal 0. Let us suppose that it is decidable weather a real is rational. Then the procedure $\Pi_{\alpha,\beta,\mathcal{W}}(M)$ where $\mathcal{W}$ is the poperty of being rational, decides wheather M is in $F^{\shortparallel}\omega$. This is, however, impossible.

(•) (r is algebraic of degree n)

Let $r \overset{\text{DF}}{=} \sqrt[n]{2}-1$. Of course r is algebraic and its degree is n. Let $\alpha \overset{\text{DF}}{=} \delta_2(r)$ and let $\beta \overset{\text{DF}}{=} \delta_2(\sqrt[n+1]{2}-1)$. Then $[\![H(\alpha,\beta,M)]\!]_2$ is algebraic with degree

n iff $M \in H^{||}\omega$. Hence $\Pi_{\alpha,\beta,\mathcal{W}}$ decides wheather M is in $F^{||}\omega$, which is impossible.

(•) (r contains in its expansion the digit 1)
Let α be constantly equal 1, and let β be constantly equal 0. Then the number $r \stackrel{\text{DF}}{=} [\![H(\alpha,\beta,M)]\!]_2$ contains a digit 1 iff $M \in F^{||}\omega$. Hence decidability of this property would yield decidability of $F^{||}\omega$. Also r equals 0 iff $M \notin F^{||}\omega$. Hence, we obtain undecidability of being equal 0.

■

As a corollary we have:

Corollary 2.4 It is not decidable if a real r is in the Cantor set.

Proof. The Cantor set consists of only these reals which have no digit 2 in their triadic expansions. ■

Now, we will prove Theorem 1.11.

Proof. In the below α and β will denote sequences of real numbers.

(•) (χ is convergent)
Let α be a sequence of reals given by $\alpha_k \stackrel{\text{DF}}{=} \frac{(1+(-1))^k}{2}$, $k \in \omega$ and let β be constantly equal 0. Then $H(\alpha,\beta,M)$ is convergent iff $M \notin F^{||}\omega$.

(•) (χ) has no accumulation points)
Let α be given by $\alpha_k \stackrel{\text{DF}}{=} k$ and let β be constantly 0. Then $H(\alpha,\beta,M)$ has an accumulation point iff $M \notin F^{||}\omega$.

(•) r is an accumulation point of χ)
This immediately follows from the latter. There 0 is an accumulation point of $H(\alpha,\beta,M)$ iff $M \notin F^{||}\omega$.

(•) (χ is monotone, constant)
α and β as above. Then $H(\beta,\alpha,M)$ is strictly monotone iff $M \notin F^{||}\omega$. Also $H(\alpha,\beta,M)$ is constant iff $M \notin F^{||}\omega$. ■

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