Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VI, Częstochowa 1999

The identity problem for the variety of all metabelian groups

Martin Kuřil

1 Introduction

This contribution will present an application of results from [1] and [2] in the group theory.

A group G is called metabelian if there is a normal subgroup N of G such that

(i) N is abelian

(ii) G/N is abelian.

A group G is metabelian if and only if it satisfies the identity

$$x_1x_2x_1^{-1}x_2^{-1}x_3x_4x_3^{-1}x_4^{-1} = x_3x_4x_3^{-1}x_4^{-1}x_1x_2x_1^{-1}x_2^{-1}.$$

A unary semigroup is an algebra $S = (S, \cdot, ')$ with an associative multiplication and with a unary operation $'$.

Let Y be a non-empty set. We add new symbols (and)$'$ to the set Y and obtain a set $Y_0 = Y \cup \{(,)'\}$. Let us denote the free semigroup on an alphabet A by A^+. Let $U(Y)$ be the smallest one among the subsets T in Y_0^+ which satisfy

(i) $Y \subseteq T$

(ii) $u, v \in T$ implies $uv \in T$

(iii) $u \in T$ implies $(u)' \in T$.

The set $U(Y)$ can be considered as a unary semigroup with a binary operation given by the concatenation of words and with a unary operation $' : U(Y) \to U(Y)$ given by $u \mapsto (u)'$. The unary semigroup $U(Y)$ is the free unary semigroup on the set Y.

In order to reduce the number of brackets in formulas, we will omit them if it causes no confusion. For example, we will often write u' instead of $(u)'$.

The set of all fully invariant congruences on the unary semigroup $U(Y)$ will be denoted by $FICU(Y)$. We will use the symbol $\leftrightarrow$ for the well-known one-to-one correspondence between all varieties of unary semigroups and all fully invariant congruences on the free unary semigroup $U(Y)$ provided that Y is an infinite set.

We adopt the following notations for classes of unary semigroups:
G – the class of all groups;
MAB – the class of all metabelian groups;
AB – the class of all abelian groups.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables. Let $\gamma, \mu, \alpha \in FIC\,U(X)$, $\mathbf{G} \leftrightarrow \gamma, \mathbf{MAB} \leftrightarrow \mu, \mathbf{AB} \leftrightarrow \alpha$.

What does it mean to solve the identity problem for the variety of all metabelian groups? It means to give an effective description of the relation μ.

2 A solution of the identity problem

Recall the classical multiplication of group varieties. Let $\mathcal{U}, \mathcal{V}$ be group varieties. We define a new class of groups

$$\mathcal{UV} = \{G \in \mathbf{G} \mid \text{there is a normal subgroup } N \text{ of } G \text{ such that } N \in \mathcal{U}, G/N \in \mathcal{V}\}.$$

In fact, the class $\mathcal{UV}$ is again a variety. Clearly, $\mathbf{MAB} = \mathbf{AB} \cdot \mathbf{AB}$.

Given $\rho \in FIC\,U(X)$, define a new alphabet $X_\rho = U(X)/\rho \times X$.

Define a left action $*$ of $U(X)$ on $U(X_\rho)$ by

$$\begin{aligned} u * (v\rho, x) &= (uv\rho, x) \\ u * ab &= (u * a)(u * b) \\ u * a' &= (u * a)' \end{aligned}$$

for $u, v \in U(X), x \in X, a, b \in U(X_\rho)$.

Now, let $\rho \in FIC\,U(X), \rho \supseteq \gamma$. Define

$$\pi_\rho : U(X) \to U(X_\rho)$$

by

$$\begin{aligned} \pi_\rho(x) &= (1, x) \\ \pi_\rho(uv) &= \pi_\rho(u)(u * \pi_\rho(v)) \\ \pi_\rho(u') &= u' * (\pi_\rho(u))' \end{aligned}$$

where $x \in X, u, v \in U(X)$ and 1 stands for the identity element of the group $U(X)/\rho$.

Theorem. *Let $\rho \in FICU(X), \sigma \in FICU(X_\rho)$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathbf{G}$ be varieties such that $\mathcal{U} \leftrightarrow \sigma, \mathcal{V} \leftrightarrow \rho$. Denote by $\sigma\rho$ the fully invariant congruence on $U(X)$ corresponding to the variety $\mathcal{UV}$. Then*

$$u(\sigma\rho)v \Longleftrightarrow u\rho v \text{ and } \pi_\rho(u)\sigma\pi_\rho(v)$$

(for all $u, v \in U(X)$).

Proof. The assertion follows immediately from [1] 4.13 and [2] 8.2.

Now, we can formulate our solution of the identity problem for the variety of all metabelian groups. Let $\alpha' \in FICU(X_\alpha), \mathbf{AB} \leftrightarrow \alpha'$.

Corollary. *For any $u, v \in U(X)$ we have*

$$u\mu v \Longleftrightarrow u\alpha v \text{ and } \pi_\alpha(u)\alpha'\pi_\alpha(v).$$

References

[1] M.Kuřil, *A multiplication of e-varieties of regular E-solid semigroups by inverse semigroup varieties*, Archivum mathematicum 33 (1997), 279 – 299

[2] M.Kuřil, *Násobení e-variet regulárních pologrup*, dissertation, Faculty of Science, Masaryk University, Brno 1995

Department of Mathematics
J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem
Czech Republic
e-mail: kurilm@pf.ujep.cz