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Intuitionistic Models of Arithmetic

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A *Kripke structure* $\mathcal{K} = (\mathcal{K}, \leq, \{\mathcal{M}_\alpha : \alpha \in \mathcal{K}\})$ for a first-order theory in the language L consists of a non-empty set K of *nodes*, partially ordered by $\leq$ —which constitute the *frame* of the model— and a set of classical structures $\mathcal{M}_\alpha$ for the language L satisfying the condition: if $\alpha \leq \beta$ then $\mathcal{M}_\alpha$ is a submodel of $\mathcal{M}_\beta$. The structures $\mathcal{M}_\alpha$ will be called the *worlds* of the Kripke structure $\mathcal{K}$.

It is convenient to consider with each world $\mathcal{M}_\alpha$ the language L_α which results in expanding the language L with constant symbols for every element of the domain $|\mathcal{M}_\alpha|$ of the world $\mathcal{M}_\alpha$. We will use the same symbol for the element of $|\mathcal{M}_\alpha|$ and the constant symbol in L_α corresponding to it.

In every Kripke structure we observe the interplay of two notions of truth: the classical truth (or satisfaction) in a model $\mathcal{M}_\alpha$ and the intuitionistic forcing at the node α. As usual, we will write $\mathcal{M}_\alpha \models A$ if A is classically true in the model $\mathcal{M}_\alpha$ and $\alpha \vdash A$ if A is forced at the node α. The forcing relation is defined inductively on the complexity of the formula A as follows:

- If A is atomic, $\alpha \Vdash A$ iff $\mathcal{M}_\alpha \models A$; in particular, $\alpha \nVdash \bot$.
- $\alpha \Vdash B \vee C$ iff $\alpha \Vdash B$ or $\alpha \Vdash C$.
- $\alpha \Vdash B \wedge C$ iff $\alpha \Vdash B$ and $\alpha \Vdash C$.
- $\alpha \Vdash B \rightarrow C$ iff for every $\beta \geq \alpha$, if $\beta \Vdash B$ then $\beta \Vdash C$.
- $\alpha \Vdash \exists x Bx$ iff there is some c in $|\mathcal{M}_\alpha|$ such that $\alpha \Vdash Bc$
- $\alpha \Vdash \forall x Bx$ iff for every $\beta \geq \alpha$ and for every c in $|\mathcal{M}_\beta|$, $\beta \Vdash Bc$

As usual, we say that the formula A is *valid* in the Kripke structure $\mathcal{K}$, denoted $\mathcal{K} \Vdash \mathcal{A}$, if $\alpha \Vdash A$, for all nodes $\alpha \in K$. A Kripke structure $\mathcal{K}$ will be called a *model* of the (intuitionistic) theory T, denoted by $\mathcal{K} \Vdash \mathcal{T}$, if all sentences provable in T are forced in $\mathcal{K}$.

Notice that if $\alpha \Vdash A$ and $\beta \geq \alpha$ then also $\beta \Vdash A$. This immediate consequence of the definition of forcing can be easily proved by the induction on the complexity of A. On the other hand, it is clear that $\mathcal{M}_\alpha \models A$ does not necessarily imply $\mathcal{M}_\beta \models A$ for $\alpha \leq \beta$ and, in general, there is no reason for a (complex) formula A forced at α to be satisfied in $\mathcal{M}_\alpha$ and vice versa. In this situation it seems to be natural to ask about the (classical) theories corresponding to the worlds $\mathcal{M}_\alpha$ of a given Kripke model of the particular (intuitionistic) theory T.

In this paper we consider Heyting Arithmetic (HA) which can be regarded as the intuitionistic version of Peano Arithmetic (PA). Usually HA is considered as a theory in the language containing the constant symbol 0 and a function symbol for every primitive recursive function. The axioms of HA are thus the specific axioms of Peano Arithmetic (PA) in the considered language —in particular among the axioms of HA there are the definitions of all primitive recursive functions— plus the usual axiomatization of Intuitionistic Predicate Logic. It is clear that PA can be obtained by adding to HA e.g. the Principle of Excluded Middle, $\forall x(A \vee \neg A)$.

Another theory which will be considered in this paper is Primitive Recursive Arithmetic (PRA), the fragment of HA in the language of Δ_0 formulae. It is known that PRA proves $A \vee \neg A$, so PRA is a common fragment of both HA and PA.

Decidability of formulae of PRA entails, in terms of Kripke models, the following well-known fact:

Theorem 1 *Let $\mathcal{K}$ be a Kripke structure for the language of arithmetic and let $\mathcal{K} \Vdash$ PRA. Then, for every $\alpha \in K$ and for every Δ_0-formula A:*

$$\alpha \Vdash A \iff \mathcal{M}_\alpha \models A.$$

This fact implies that the worlds of a model of PRA are ordered not merely by a submodel relation (consistent with the ordering of the frame) but in fact by the relation of being an elementary submodel with respect to Δ_0 formulae. More precisely,

Theorem 2 *If $\mathcal{K}$ is a model of* PRA *then for every nodes α, β of K, if $\alpha \leq \beta$ then $\mathcal{M}_\alpha \prec_{\Delta_0} \mathcal{M}_\beta$.*

From Theorem 1 we derive the following characterization of the worlds of models of PRA:

Theorem 3 *A Kripke structure $\mathcal{K}$ is a model of* PRA *iff all the worlds of $\mathcal{K}$ are models of* PRA.

In case of models of HA one could expect a result analogous to the above characterization. Namely, it would seem very natural that every Kripke model whose all worlds are models of PA would be a model of HA. And conversely, one could expect that all the worlds of a Kripke model of HA would be the (classical) models of PA. We will briefly present some results concerning this two conjectures.

We say that a world $\mathcal{M}_\alpha$ of a Kripke model $\mathcal{K}$ is a *Peano world* iff $\mathcal{M}_\alpha \models \mathsf{PA}$ and that the model $\mathcal{K}$ is *locally* PA iff all its worlds are Peano. It is clear that if $\mathcal{K}$ is locally Peano then $\mathcal{K} \Vdash \mathsf{PRA}$.

In [1] it is shown that there is a Kripke model $\mathcal{K}$ which is locally PA and $\mathcal{K}$ is not a model of HA. Thus the first conjecture has the negative solution. In his paper S. Buss constructs a locally PA model which does not force the instance of the induction schema for a Π_1-formula (which will be denoted by $I(\Pi_1)$) and hence it cannot be a model of HA. The model constructed by S. Buss is infinite (its frame is the set of all positive integers with the natural ordering) and its worlds do not satisfy the condition $\mathcal{M}_\alpha \prec_{\Sigma_1} \mathcal{M}_\beta$ for $\alpha \leq \beta$. One can ask whether these conditions are optimal. In order to answer this question we prove the following results.

Theorem 4 *For every finite, locally* PA *model* $\mathcal{K}$, $\mathcal{K} \Vdash \mathcal{I}(\diamond_\infty)$.

Concerning the second property of the Buss' model we prove:

Theorem 5 *Let* $\mathcal{K}$ *be a model consisting of worlds such that* $\mathcal{M}_\alpha \prec_{\Sigma_1} \mathcal{M}_\beta$ *for* $\alpha \leq \beta$*. Then, for every* $\alpha \in K$*, if* $\mathcal{M}_\alpha \models I(\Pi_1)$ *then* $\alpha \Vdash I(\Pi_1)$*. In particular, if* $\mathcal{K}$ *is a locally* PA *model whose worlds are ordered by* $\prec_{\Sigma_1}$ *then* $\mathcal{K}$ *forces induction schema for all* Π_1 *formulae.*

However, it turns out that the assumption about Σ_1-equivalence of the appropriate worlds turns out to be very strong when the models of HA are considered. Indeed, we prove

Theorem 6 *Let* $\mathcal{K}$ *be a model of* HA *such that for every* $\alpha, \beta \in K$ *if* $\alpha \leq \beta$ *then* $\mathcal{M}_\alpha \prec_{\Sigma_1} \mathcal{M}_\beta$*. Then for every* $Ax \in \Sigma_1$*:*

$$\mathcal{K} \Vdash \neg\neg\exists x Ax \ to \exists x Ax,$$

i.e. $\mathcal{K}$ *validates the Markov's principle for* Σ_1 *formulae (which is unprovable in* HA*).*

Hence, if $\mathcal{K} \Vdash \mathsf{HA}$ and additionally for every $\alpha, \beta \in K$ the condition $\alpha \leq \beta$ implies $\mathcal{M}_\alpha \prec_{\Sigma_1} \mathcal{M}_\beta$, then $\mathsf{HA} \not\subseteq Th(\mathcal{K})$. So the class of all models whose appropriate worlds are Σ_1-equivalent cannot adequate with respect to HA.

Turning to the second conjecture, we note that it is still an open problem whether every model of HA is locally PA. However, in [2] it is shown that if $\mathcal{K}$ is finite model of HA then $\mathcal{K}$ is locally PA. This result is generalized in [3] for all models over conversely well-founded frames and models of HA over the frame $(\omega, \leq)$. In this paper we will present another partial results concerning this conjecture.

Let $\mathcal{K} = (\mathcal{K}, \leq, \{\mathcal{M}_\alpha : \alpha \in \mathcal{K}\})$ be a Kripke structure. We say that a node $\alpha \in K$ is *classical* if $\alpha \Vdash \forall \overline{x}(A \vee \neg A)$ for every formula A and every string of variables $\overline{x}$. It is easy to see that all maximal nodes are classical. Moreover, if α is classical then every $\beta \geq \alpha$ is also classical. We have the following characterization of the classical nodes:

Theorem 7 *A node α of the Kripke models $\mathcal{K}$ is classical iff for every A*

$$\alpha \Vdash A \Longleftrightarrow \mathcal{M}_\alpha \models A.$$

In particular, if α is a classical node of a Kripke model of HA *then $\mathcal{M}_\alpha \models$* PA.

By Theorem 2, in every Kripke model of PRA a world $\mathcal{M}_\alpha$ is an elementary submodel of $\mathcal{M}_\beta$ with respect to Δ_0-formula provided $\alpha \leq \beta$. One can ask what happens if we consider models in which all the appropriate structures are elementary submodels of each other with respect to all the formulae. In other words, we can ask what happens if we require the appropriate worlds to be not merely submodels but *elementary* submodels. In this case we prove the following

Theorem 8 *Let $\mathcal{K}$ be a model in which for every $\alpha, \beta \in K$ if $\alpha \leq \beta$ then $\mathcal{M}_\alpha \prec \mathcal{M}_\beta$. Then for every $\alpha \in K$ and every sentence A:*

$$\alpha \Vdash A \quad \Longleftrightarrow \quad \mathcal{M}_\alpha \models A.$$

From Theorem 8 it follows that if $\mathcal{K}$ is a model in which all worlds are elementarily equivalent then all nodes of $\mathcal{K}$ are classical. Hence, by Theorem 7 we get the following result:

Theorem 9 *Let $\mathcal{K}$ be a model in which all worlds are elementarily equivalent. Then $\mathcal{K} \Vdash$* HA *iff $\mathcal{K}$ is locally* PA. *Moreover, if $\mathcal{K}$ is locally* PA *then $\mathcal{K} \Vdash$* PA.

Theorem 9 gives us an answer in a case of very special models. However, in the general case the problem is still open. Moreover, it is still unknown whether any model of HA must contain at least one Peano world. It seems that even the solution to the latter problem would provide us with a great deal of information about intuitionistic models of arithmetic.

References

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