Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VI, Częstochowa 1999

Cosserat continua

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The classical continuum theory of solids is based upon the assumption that each small particle behaves like a single material point and ignores the relative motions of constituent parts of this particle. The internal logic of the development of mechanics of continuous media, as well as the growing field of applications of the theory, has led to the study of media with microstructure, for example, the microstructure theory of Mindlin [8], the micromorphic theory of Eringen and Suhubi [5], the director theory of Toupin [19], the multipolar theory of Green and Rivlin [6], etc. The motion of Cosserat continuum is determined both by the displacement field $\mathbf{u}$ and by the rotation field $\boldsymbol{\varphi}$ independent on it, which causes the appearance of couple–stresses $\boldsymbol{\tau}$ alongside the usual stresses $\boldsymbol{\sigma}$ [3].

A convinient device for studying the Cosserat continua is the motor calculus. The algebra of motors was developed by Mises [9, 10], the differential operators for three–dimensional motors were introduced in [11, 12, 16, 17], the motor analysis for a Cosserat surface and a Cosserat line was developed by the author [14].

A motor is the ordered pair of two vectors: the 'moment vector' and the 'ordinary (force) vector'. The underlined letters explain the origin of a term 'motor'.

For example, a force field in a rigid body can be reduced to a force vector $\mathbf{F}(P)$ at a point P and to a moment vector $\mathbf{C}(P)$ at the same point

$$\mathbf{M}(P) = \begin{pmatrix} \mathbf{F}(P) \\ \mathbf{C}(P) \end{pmatrix}. \tag{1}$$

Changing a reduction point we obtain

$$\mathbf{M}(Q) = \begin{pmatrix} \mathbf{F}(Q) \\ \mathbf{C}(Q) \end{pmatrix} = \begin{pmatrix} \mathbf{F}(P) \\ \mathbf{C}(P) + \mathbf{F}(P) \times \overrightarrow{QP} \end{pmatrix}. \tag{2}$$

For a given point P of a rigid body infinitesimal translation and infinitesimal rotation are described by the translation vector $\mathbf{u}(P)$ and the rotation vector $\varphi(P)$ forming a motor

$$\begin{pmatrix} \varphi(P) \\ \mathbf{u}(P) \end{pmatrix}. \tag{3}$$

For another point Q

$$\begin{pmatrix} \varphi(Q) \\ \mathbf{u}(Q) \end{pmatrix} = \begin{pmatrix} \varphi(P) \\ \mathbf{u}(P) + \varphi(P) \times \overrightarrow{QP} \end{pmatrix}. \tag{4}$$

Equation (2) (or (4)) reflects the essential feature of the motor. Thus, a motor is the ordered pair of two vectors which change according to a rule (2) when changing a reduction point.

Introducing the three-dimensional del-operator

$$\boldsymbol{\nabla} = \mathbf{e}^k \frac{\partial}{\partial x^k}, \quad k = 1, 2, 3, \tag{5}$$

where x^k are the curvilinear coordinates, $\mathbf{e}^k$ are vectors of the local basis, we use the following invariant notations for the gradient of a motor field $\binom{\mathbf{V}}{\mathbf{W}}$, the divergence and the curl of a motor-tensor field $\binom{\mathbf{Q}}{\mathbf{R}}$:

$$\boldsymbol{\nabla} \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla} \mathbf{V} \\ \boldsymbol{\nabla} \mathbf{W} - (\mathbf{V} \times \mathbf{g})^T \end{pmatrix}, \tag{6}$$

$$\boldsymbol{\nabla} \cdot \begin{pmatrix} \mathbf{Q} \\ \mathbf{R} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla} \cdot \mathbf{Q} \\ \boldsymbol{\nabla} \cdot \mathbf{R} - (\mathbf{Q} \overset{\times}{\cdot} \mathbf{g})^T \end{pmatrix}, \tag{7}$$

$$\boldsymbol{\nabla} \times \begin{pmatrix} \mathbf{Q} \\ \mathbf{R} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla} \times \mathbf{Q} \\ \boldsymbol{\nabla} \times \mathbf{R} - (\mathbf{Q} \overset{\times}{\times} \mathbf{g})^T \end{pmatrix}. \tag{8}$$

We use the following order of operations $\overset{\times}{\cdot}$ and $\overset{\times}{\times}$ in (7) and (8): the cross product of the neighbouring basis vectors of tensor $\mathbf{Q}$ and the metric tensor $\mathbf{g}$; the permutation of the second and the third basis vectors in the product ($\mathbf{Q}^T$ denotes the transpose of tensor $\mathbf{Q}$); the scalar or cross product of the first and the second basis vectors.

The repeated action of the del-operator leads to the motor analogue [17] of the well-known relations

$$\boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \end{pmatrix} \right] = 0, \tag{9}$$

$$\boldsymbol{\nabla} \cdot \left[\boldsymbol{\nabla} \times \begin{pmatrix} \mathbf{Q} \\ \mathbf{R} \end{pmatrix} \right] = 0. \tag{10}$$

The Gauss–Ostrogradski formula is also fulfilled for motors [11]

$$\int_V \nabla \cdot \begin{pmatrix} \mathbf{Q} \\ \mathbf{R} \end{pmatrix} \mathrm{d}V = \int_A \mathbf{n} \cdot \begin{pmatrix} \mathbf{Q} \\ \mathbf{R} \end{pmatrix} \mathrm{d}A, \tag{11}$$

where $\mathbf{n}$ denotes the unit normal to a surface A which is a boundary of a volume V. It is natural that all the terms in (11) must be written for arbitrary (non–zero) point of reduction.

Substituting the first fundamental tensor of a surface $\mathbf{a}$ for the metric tensor $\mathbf{g}$, we define the following differential operators for motors given at two–dimensional surface embedded into a three–dimensional space

$$\nabla_\Sigma \begin{pmatrix} \mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma \end{pmatrix} = \begin{pmatrix} \nabla \mathbf{V}_\Sigma \\ \nabla \mathbf{W}_\Sigma - (\mathbf{V}_\Sigma \times \mathbf{a})^T \end{pmatrix}, \tag{12}$$

$$\nabla_\Sigma \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} = \begin{pmatrix} \nabla_\Sigma \cdot \mathbf{Q}_\Sigma \\ \nabla_\Sigma \cdot \mathbf{R}_\Sigma - (\mathbf{Q}_\Sigma \overset{\times}{\cdot} \mathbf{a})^T \end{pmatrix}, \tag{13}$$

$$\nabla_\Sigma \times \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} = \begin{pmatrix} \nabla_\Sigma \times \mathbf{Q}_\Sigma \\ \nabla_\Sigma \times \mathbf{R}_\Sigma - (\mathbf{Q}_\Sigma \overset{\times}{\times} \mathbf{a})^T \end{pmatrix}, \tag{14}$$

where ∇_Σ is the surface del–operator

$$\nabla_\Sigma = \mathbf{a}^\alpha \frac{\partial}{\partial y^\alpha}, \qquad \alpha = 1, 2 \tag{15}$$

with y^α being the surface curvilinear coordinates, $\mathbf{a}^\alpha$ vectors of the local surface basis.

The equations analogous to equations (9) and (10)

$$\nabla_\Sigma \times \left[\nabla_\Sigma \begin{pmatrix} \mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma \end{pmatrix} \right] = \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \nabla_\Sigma \begin{pmatrix} \mathbf{V}_\Sigma \\ \mathbf{W}_\Sigma \end{pmatrix}, \tag{16}$$

$$\nabla_\Sigma \cdot \left[\nabla_\Sigma \times \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right] = \nabla_\Sigma \cdot \left[\boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \nabla_\Sigma \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right] - 2H \mathbf{n} \cdot \left[\nabla_\Sigma \times \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \right] \tag{17}$$

and the surface analogue of (11)

$$\int_\Sigma \nabla_\Sigma \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}\Sigma = \int_L \mathbf{N} \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}L - \int_\Sigma 2H \mathbf{n} \cdot \begin{pmatrix} \mathbf{Q}_\Sigma \\ \mathbf{R}_\Sigma \end{pmatrix} \mathrm{d}\Sigma \tag{18}$$

are fulfilled, where $\boldsymbol{\epsilon}_\Sigma$ is the surface alternating tensor, $\mathbf{b}$ denotes the second fundamental tensor of the surface, $H = \frac{1}{2} b^\alpha_\alpha$ is the mean curvature. The closed curve L is the boundary of the surface Σ and $\mathbf{N}$ is the outward unit vector normal to L and tangential to Σ.

In a case of line embedded in a three-dimensional space we have the one-dimensional del-operator

$$\nabla_L = \boldsymbol{\lambda}\frac{\partial}{\partial s}, \tag{19}$$

where s denotes the length of a curve and $\boldsymbol{\lambda}$ is the unit tangential vector.

For the line differential operators

$$\nabla_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix} = \begin{pmatrix} \nabla \mathbf{V}_L \\ \nabla \mathbf{W}_L - (\mathbf{V}_L \times \boldsymbol{\lambda} \otimes \boldsymbol{\lambda})^T \end{pmatrix}, \tag{20}$$

$$\nabla_L \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} = \begin{pmatrix} \nabla_L \cdot \mathbf{Q}_L \\ \nabla_L \cdot \mathbf{R}_L - (\mathbf{Q}_L \overset{\times}{\cdot} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda})^T \end{pmatrix}, \tag{21}$$

$$\nabla_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} = \begin{pmatrix} \nabla_L \times \mathbf{Q}_L \\ \nabla_L \times \mathbf{R}_L - (\mathbf{Q}_L \overset{\times}{\times} \boldsymbol{\lambda} \otimes \boldsymbol{\lambda})^T \end{pmatrix} \tag{22}$$

the following formulae

$$\nabla_L \times \left[\nabla_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix} \right] = k\boldsymbol{\nu} \otimes \boldsymbol{\lambda} \cdot \left[\nabla_L \begin{pmatrix} \mathbf{V}_L \\ \mathbf{W}_L \end{pmatrix} \right], \tag{23}$$

$$\nabla_L \cdot \left[\nabla_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right] = -k\boldsymbol{\tau} \cdot \left[\nabla_L \times \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right] \tag{24}$$

and the one-dimensional analogue of the Gauss-Ostrogradski formula

$$\int_L \nabla_L \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \mathrm{d}L = \left[\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \right]_-^+ - \int_L k\boldsymbol{\tau} \cdot \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{R}_L \end{pmatrix} \mathrm{d}L \tag{25}$$

hold with k the first curvature, $\boldsymbol{\tau}$ the principal normal, $\boldsymbol{\nu}$ the binormal to a curve ($\boldsymbol{\lambda}$, $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ is the Frenet trihedron).

Now we shall write the basic equations for Cosserat continua of various dimensions in motor recording (see also [7, 15]).

Three-dimensional Cosserat continuum

The equation of equilibrium

$$\nabla \cdot \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{pmatrix} = -\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}. \tag{26}$$

The geometrical relations

$$\begin{pmatrix}\boldsymbol{\kappa}\\ \boldsymbol{\gamma}\end{pmatrix} = \boldsymbol{\nabla}\begin{pmatrix}\boldsymbol{\varphi}\\ \mathbf{u}\end{pmatrix}. \tag{27}$$

The condition of compatibility

$$\begin{pmatrix}\boldsymbol{\theta}\\ \boldsymbol{\alpha}\end{pmatrix} = -\boldsymbol{\nabla}\begin{pmatrix}\boldsymbol{\kappa}^p\\ \boldsymbol{\gamma}^p\end{pmatrix}. \tag{28}$$

The stress–strain relation

$$\begin{pmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{C}\\ \mathbf{D} & \mathbf{0}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\kappa}\\ \boldsymbol{\gamma}\end{pmatrix}. \tag{29}$$

Defect currents

$$\begin{pmatrix}\mathbf{I}\\ \mathbf{J}\end{pmatrix} = \begin{pmatrix}\dot{\boldsymbol{\kappa}}^p\\ \dot{\boldsymbol{\gamma}}^p\end{pmatrix} - \boldsymbol{\nabla}\begin{pmatrix}\mathbf{w}^p\\ \mathbf{v}^p\end{pmatrix}. \tag{30}$$

The kinematical equation

$$\begin{pmatrix}\dot{\boldsymbol{\theta}}\\ \dot{\boldsymbol{\alpha}}\end{pmatrix} = -\boldsymbol{\nabla}\times\begin{pmatrix}\mathbf{I}\\ \mathbf{J}\end{pmatrix}. \tag{31}$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are the volume force and the volume couple, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the stress tensor and the couple–stress tensor

$$\boldsymbol{\sigma} = \sigma^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{\mu} = \mu^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{32}$$

$\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are the strain tensor and the bend–twist tensor

$$\boldsymbol{\gamma} = \gamma^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \boldsymbol{\kappa} = \kappa^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{33}$$

$\boldsymbol{\gamma}^p$ and $\boldsymbol{\kappa}^p$ are the plastic strain and plastic bend–twist tensors, $\mathbf{v}^p$ and $\mathbf{w}^p$ are the velocities of plastic displacement and plastic rotation, $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ are the densities of dislocations and disclinations,

$$\boldsymbol{\alpha} = \alpha^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{\theta} = \theta^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{34}$$

$\mathbf{J}$ anf $\mathbf{I}$ are the dislocation and disclination currents

$$\mathbf{J} = J^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{I} = I^{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{35}$$

$\mathbf{C}$ and $\mathbf{D}$ denote the isotropic fourth–order tensors

$$\mathbf{C} = C_{ikmp}\mathbf{e}^i \otimes \mathbf{e}^k \otimes \mathbf{e}^m \otimes \mathbf{e}^p \qquad \mathbf{D} = D_{ikmp}\mathbf{e}^i \otimes \mathbf{e}^k \otimes \mathbf{e}^m \otimes \mathbf{e}^p,$$

$$C_{ikmp} = \lambda g_{ik} g_{mp} + \mu(g_{im} g_{kp} + g_{ip} g_{km}) + \alpha(g_{im} g_{kp} - g_{ip} g_{km}),$$

$$D_{ikmp} = \beta g_{ik} g_{mp} + \gamma(g_{im} g_{kp} + g_{ip} g_{km}) + \epsilon(g_{im} g_{kp} - g_{ip} g_{km}),$$

where λ, μ, α, β, γ, ϵ are material constants. A dot denotes the time derivative.

Two–dimensional Cosserat continuum

The equation of equilibrium

$$\nabla_\Sigma \cdot \begin{pmatrix} \boldsymbol{\sigma}_\Sigma \\ \boldsymbol{\mu}_\Sigma \end{pmatrix} = - \begin{pmatrix} \mathbf{X}_\Sigma \\ \mathbf{Y}_\Sigma \end{pmatrix}. \tag{36}$$

The geometrical relations

$$\begin{pmatrix} \boldsymbol{\kappa}_\Sigma \\ \boldsymbol{\gamma}_\Sigma \end{pmatrix} = \nabla_\Sigma \begin{pmatrix} \boldsymbol{\varphi}_\Sigma \\ \mathbf{u}_\Sigma \end{pmatrix}. \tag{37}$$

The condition of compatibility

$$\begin{pmatrix} \boldsymbol{\theta}_\Sigma \\ \boldsymbol{\alpha}_\Sigma \end{pmatrix} = -\nabla_\Sigma \begin{pmatrix} \boldsymbol{\kappa}^p_\Sigma \\ \boldsymbol{\gamma}^p_\Sigma \end{pmatrix} + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix} \boldsymbol{\kappa}^p_\Sigma \\ \boldsymbol{\gamma}^p_\Sigma \end{pmatrix}. \tag{38}$$

The stress–strain relation

$$\begin{pmatrix} \boldsymbol{\sigma}_\Sigma \\ \boldsymbol{\mu}_\Sigma \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{C}_\Sigma \\ \mathbf{D}_\Sigma & \mathbf{0} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\kappa}_\Sigma \\ \boldsymbol{\gamma}_\Sigma \end{pmatrix}. \tag{39}$$

Defect currents

$$\begin{pmatrix} \mathbf{I}_\Sigma \\ \mathbf{J}_\Sigma \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{\kappa}}^p_\Sigma \\ \dot{\boldsymbol{\gamma}}^p_\Sigma \end{pmatrix} - \nabla_\Sigma \begin{pmatrix} \mathbf{w}^p_\Sigma \\ \mathbf{v}^p_\Sigma \end{pmatrix}. \tag{40}$$

The kinematical equation

$$\begin{pmatrix} \dot{\boldsymbol{\theta}}_\Sigma \\ \dot{\boldsymbol{\alpha}}_\Sigma \end{pmatrix} = -\nabla_\Sigma \times \begin{pmatrix} \mathbf{I}_\Sigma \\ \mathbf{J}_\Sigma \end{pmatrix} + \boldsymbol{\epsilon}_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix} \mathbf{I}_\Sigma \\ \mathbf{J}_\Sigma \end{pmatrix}. \tag{41}$$

Vectors $\mathbf{X}_\Sigma$ and $\mathbf{Y}_\Sigma$ include an external loading, body forces and body couples.

In the local basis formed by the tangential vectors $\mathbf{a}_\alpha$, $(\alpha = 1, 2)$ and the unit normal $\mathbf{n}$ the introduced tensors have the following structure

$$\boldsymbol{\sigma}_\Sigma = \sigma^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \sigma^{\alpha n} \mathbf{a}_\alpha \otimes \mathbf{n},$$

$$\boldsymbol{\mu}_\Sigma = \mu^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \mu^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \tag{42}$$

$$\boldsymbol{\gamma}_\Sigma = \gamma^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \gamma^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n},$$

$$\boldsymbol{\kappa}_\Sigma = \kappa^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + \kappa^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}, \tag{43}$$

$$\boldsymbol{\alpha}_\Sigma = \alpha^{n\beta}\mathbf{n} \otimes \mathbf{a}_\beta + \alpha^{nn}\mathbf{n} \otimes \mathbf{n},$$

$$\boldsymbol{\theta}_\Sigma = \theta^{n\beta}\mathbf{n} \otimes \mathbf{a}_\beta + \theta^{nn}\mathbf{n} \otimes \mathbf{n}, \tag{44}$$

$$\mathbf{J}_\Sigma = J^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + J^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n},$$

$$\mathbf{I}_\Sigma = I^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + I^{\alpha n}\mathbf{a}_\alpha \otimes \mathbf{n}. \tag{45}$$

In the same basis tensors $\mathbf{C}_\Sigma$ and $\mathbf{D}_\Sigma$ have the components

$$\mathbf{C}_\Sigma = C_{\alpha\beta\gamma\delta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \otimes \mathbf{a}^\gamma \otimes \mathbf{a}^\delta + C_{\alpha n\beta n}\mathbf{a}^\alpha \otimes \mathbf{n} \otimes \mathbf{a}^\beta \otimes \mathbf{n},$$

$$\mathbf{D}_\Sigma = D_{\alpha\beta\gamma\delta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \otimes \mathbf{a}^\gamma \otimes \mathbf{a}^\delta + D_{\alpha n\beta n}\mathbf{a}^\alpha \otimes \mathbf{n} \otimes \mathbf{a}^{abeta} \otimes \mathbf{n},$$

$$C_{\alpha\beta\gamma\delta} = \lambda_\Sigma a_{\alpha\beta}a_{\gamma\delta} + \mu_\Sigma(a_{\alpha\gamma}a_{\beta\delta} + a_{\alpha\delta}a_{\beta\gamma}) + \alpha_\Sigma(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma}),$$

$$D_{\alpha\beta\gamma\delta} = \beta_\Sigma a_{\alpha\beta}a_{\gamma\delta} + \gamma_\Sigma(a_{\alpha\gamma}a_{\beta\delta} + a_{\alpha\delta}a_{\beta\gamma}) + \epsilon_\Sigma(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma}),$$

$$C_{\alpha n\beta n} = 2\mu_\Sigma^\perp a_{\alpha\beta},$$

$$D_{\alpha n\beta n} = 2\gamma_\Sigma^\perp a_{\alpha\beta},$$

where λ_Σ, μ_Σ, $\mu_\Sigma^\perp$, α_Σ, β_Σ, γ_Σ, $\gamma_\Sigma^\perp$, ϵ_Σ are material constants.

Equations (36)–(41) can be considered as generalized equations for the shell theory.

One–dimensional Cosserat continuum

The equation of equilibrium

$$\boldsymbol{\nabla}_L \cdot \begin{pmatrix} \boldsymbol{\sigma}_L \\ \boldsymbol{\mu}_L \end{pmatrix} = -\begin{pmatrix} \mathbf{X}_L \\ \mathbf{Y}_L \end{pmatrix}. \tag{46}$$

The geometrical relations

$$\begin{pmatrix} \boldsymbol{\kappa}_L \\ \boldsymbol{\gamma}_L \end{pmatrix} = \boldsymbol{\nabla}_L \begin{pmatrix} \boldsymbol{\varphi}_L \\ \mathbf{u}_L \end{pmatrix}. \tag{47}$$

The stress–strain relation

$$\begin{pmatrix}\boldsymbol{\sigma}_L\\ \boldsymbol{\mu}_L\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{C}_L\\ \mathbf{D}_L & \mathbf{0}\end{pmatrix}:\begin{pmatrix}\boldsymbol{\kappa}_L\\ \boldsymbol{\gamma}_L\end{pmatrix}. \tag{48}$$

In the Frenet basis the stress tensor, the couple–stress tensor, the strain tensor and the bend–twist tensor have the following structure

$$\boldsymbol{\sigma}_L=\sigma^{\lambda\lambda}\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+\sigma^{\lambda\tau}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+\sigma^{\lambda\nu}\boldsymbol{\lambda}\otimes\boldsymbol{\nu},$$

$$\boldsymbol{\mu}_L=\mu^{\lambda\lambda}\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+\mu^{\lambda\tau}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+\mu^{\lambda\nu}\boldsymbol{\lambda}\otimes\boldsymbol{\nu}, \tag{49}$$

$$\boldsymbol{\gamma}_L=\gamma^{\lambda\lambda}\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+\gamma^{\lambda\tau}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+\gamma^{\lambda\nu}\boldsymbol{\lambda}\otimes\boldsymbol{\nu},$$

$$\boldsymbol{\kappa}_L=\kappa^{\lambda\lambda}\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+\kappa^{\lambda\tau}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+\kappa^{\lambda\nu}\boldsymbol{\lambda}\otimes\boldsymbol{\nu}. \tag{50}$$

We also present the structure of the line material constants tensors

$$\mathbf{C}_L=2\mu\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+2\mu^{\perp}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+2\mu^{\perp}\boldsymbol{\lambda}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\nu},$$

$$\mathbf{D}_L=2\gamma\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\lambda}+2\gamma^{\perp}\boldsymbol{\lambda}\otimes\boldsymbol{\tau}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\tau}+2\gamma^{\perp}\boldsymbol{\lambda}\otimes\boldsymbol{\nu}\otimes\boldsymbol{\lambda}\otimes\boldsymbol{\nu}.$$

In a case of one–dimensional Cosserat continuum we can formally write the equation of compatibility

$$\begin{pmatrix}\boldsymbol{\theta}_L\\ \boldsymbol{\alpha}_L\end{pmatrix}=-\nabla_L\begin{pmatrix}\boldsymbol{\kappa}_L^p\\ \boldsymbol{\gamma}_L^p\end{pmatrix}+k\boldsymbol{\nu}\otimes\boldsymbol{\lambda}\cdot\begin{pmatrix}\boldsymbol{\kappa}_L^p\\ \boldsymbol{\gamma}_L^p\end{pmatrix}, \tag{51}$$

but due to the structure of tensors $\boldsymbol{\gamma}_L^p$ and $\boldsymbol{\kappa}_L^p$ (50) equation (51) is satisfied identically. This means that dislocations and disclinations cannot exist in a Cosserat line.

Equations (46)–(48) can be considered as generalized equations for the rod theory.

The motor analysis is very effective in various investigations of mechanics and in the theory of defects in continua (for example, see [4, 13]). Non–Abelian generalization of the motor calculus was developed by Stumpf and Badur [18] and Badur [1] (see also [2]).

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