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From material point to Cosserat fiber bundle. Development of ideas.

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The aim of the present paper is to show the development of ideas and mathematical notions used for description of continua with microstructure.

In Euclid's *Elements* a surface is defined as something that has a length and a width but does not have a hight, a curve is something that has only a length but does not have a width and a hight, and a point is considered as something that has no parts [14].

In mechanics of continua a material point is considered, on the one hand, as a very small particle to apply methods of mathematical analysis (the notions of continuous and differentiable functions of a point); on the other hand, a material point contains a very large number of atoms to consider continuum instead of discrete lattice. Hence, ,,a point" has parts. But the classical continuum theory of solids is based upon the assumptions that each small particle behaves like a single material point and ignores the relative motions of constituent parts of this particle. In other words, the internal structure of a material point is not taken into account. Let the initial coordinate of a material point of a medium B be denoted by $\mathbf{X}$ and let this point be at a geometrical point $\mathbf{x}$ of three–dimensional Euclidean space at time t. The motion of the medium is described by the equation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

with deformation gradient

$$\frac{\partial x_k}{\partial X_K}, \qquad k = 1, 2, 3; \quad K = 1, 2, 3. \tag{2}$$

However, the results of experiments show that ,,a point" may have an internal structure which can influence the behavior of a medium (further discussion of a model of material point can be found in [20]). These facts have forced researchers to build up generalized continuum theories that take into account the internal structure of a small particle. Such a structure can be described using various methods. The microstructure theory of Mindlin

[17], the micromorphic theory of Eringen and Suhubi [9, 10], the director theory of Toupin [21], and the multipolar theory of Green and Rivlin [11, 12] should be mentioned.

Mindlin suggested that each element (point) of the material is itself a deformable continuum. These continua are fitted together smoothly, so all the simplicity of a field theory results. The macromedium is a collection of particles, with each of which is associated a micromedium. If the initial coordinate of a material point of a macromedium is $\mathbf{X}$ and if this point is at a geometrical point $\mathbf{x}$ at time t, then the displacement vector has the form

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \mathbf{u}(\mathbf{X}, t). \tag{3}$$

For a micropoint with the material Ξ and spatial $\boldsymbol{\xi}$ position vectors the microdisplacement vector is defined as

$$\mathbf{w} = \boldsymbol{\xi} - \Xi = \mathbf{w}(\mathbf{X}, \Xi, t). \tag{4}$$

The ensuing development of the theory is based on the assumptions concerning the dependence of microdisplacement $\mathbf{w}$ on the arguments $\mathbf{X}$ and Ξ.

Eringen [8] and Eringen and Suhubi [9, 10] supposed that the material particle contains N discrete micromaterial elements. The position vector of a material point in the αth microelement is expressed as

$$\mathbf{X}^{\alpha} = \mathbf{X} + \Xi^{\alpha}, \tag{5}$$

where the center of mass of macroelement has the position vector $\mathbf{X}$ and Ξ^{α} is the position of a point in the microelement relative to this center of mass. Upon the deformation of the body, because of the rearrangement and relative deformation of the microelements, we obtain the new position vector of the center of mass $\mathbf{x}$ and the new relative position vector $\boldsymbol{\xi}^{\alpha}$ of the material point. The motion of the center of mass is expressed as usually by equation (1), however the relative position vector $\boldsymbol{\xi}^{\alpha}$ depends not only on $\mathbf{X}$ but also on Ξ^{α}:

$$\boldsymbol{\xi}^{\alpha} = \boldsymbol{\xi}^{\alpha}(\mathbf{X}, \Xi^{\alpha}, t). \tag{6}$$

The displacement vector is defined as the vector that extends from $\mathbf{X}^{\alpha}$ to $\mathbf{x}^{\alpha}$

$$\mathbf{u}^{\alpha} = \mathbf{x}^{\alpha} - \mathbf{X}^{\alpha} \tag{7}$$

or

$$\mathbf{u}^{\alpha} = \mathbf{x} - \mathbf{X} + \boldsymbol{\xi}^{\alpha} - \Xi^{\alpha}, \tag{8}$$

where $\mathbf{u} = \mathbf{x} - \mathbf{X}$ is the classical displacement vector (the displacement vector of the center of mass).

A microstructure theory must lean heavily on the assumption characterizing the dependence of $\boldsymbol{\xi}^\alpha$ on Ξ^α. The basic assumption underlying the theory of Eringen and Suhubi is the axiom of affine motion according to which equation (6) is linear in Ξ^α and the motion of the particle consists of a translation, a rotation about its center of mass, and an affine deformation.

Duhem [4] had noticed that microstructure could be described as effects of direction, and suggested that materials be considered as sets of points having vectors attached to them, that is, as oriented media. Various theories based on this idea were constructed in [21, 23].

Consider a body B that is the collection of material points to each of which N vectors called directors are attached. In other words, a body B consists of material points and N directors attached to each point. The theory is valid for a greater number of directors, but usually the range 1, 2 or 3 is considered. Let the initial coordinate of a material point be denoted by $\mathbf{X}$ and the associated directors by $\mathbf{D}_a(\mathbf{X})$, $(a = 1, 2, \ldots, N)$. Upon deformation of the body, the new position of the same material point at time t will be denoted by $\mathbf{x}$ and the corresponding directors by $\mathbf{d}_a$, $(a = 1, 2, \ldots, N)$. Thus, the motion of such a generalized continuum is described by the following equations

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{9}$$

$$\mathbf{d}_a = \mathbf{d}_a(\mathbf{X}, t) \tag{10}$$

or in Cartesian coordinates

$$x_k = x_k(X_K, t), \qquad k = 1, 2, 3; \quad K = 1, 2, 3 \tag{11}$$

$$d_{ak} = d_{ak}(X_K, t), \qquad k = 1, 2, 3; \quad K = 1, 2, 3; \quad a = 1, 2, \ldots, n. \tag{12}$$

The translational velocity of a generalized material point and its microvelocities are defined by

$$\mathbf{v} \equiv \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\mathbf{X}}, \tag{13}$$

$$\boldsymbol{\nu}_a \equiv \left.\frac{\partial \mathbf{d}_a}{\partial t}\right|_{\mathbf{X}}. \tag{14}$$

Similarly, for accelerations of a material points we have

$$\mathbf{a} \equiv \left.\frac{\partial^2 \mathbf{x}}{\partial t^2}\right|_{\mathbf{X}}, \tag{15}$$

$$\mathbf{b}_a \equiv \left.\frac{\partial^2 \mathbf{d}_a}{\partial t^2}\right|_{\mathbf{X}}. \tag{16}$$

Strain gradient and microstructure gradient are

$$\frac{\partial x_k}{\partial X_K} \quad \text{and} \quad \frac{\partial d_{aK}}{\partial X_K}. \tag{17}$$

We also have the right Cauchy–Green tensor

$$a_{KL} \equiv x_{i,K} x_{i,L} \tag{18}$$

and the microstructure strain tensor

$$q_{ab} \equiv d_{ia} d_{ib}. \tag{19}$$

For example, a theory of liquid crystals proposed by Ericksen [6, 7] corresponds to a choice of only one director ($N = 1$). Liquid crystal (a substance that flows as a liquid but maintains some of the ordered structures characteristic of a crystal) has rodlike molecules whose alignment influences its material behavior. Three main categories have been recognized: nematic, cholesteric and smectic. Nematic liquid crystals consist of cigar–shaped molecules oriented with their long axes parallel. They maintain their orientation but are free to move in any direction. Cholesteric liquid crystals form in thin layers and within each layer the molecules are arranged with their long axes in the plane of the layer and parallel to each other, as a two–dimensional nematic structures. Smectic liquid crystals consist of flat layers of molecules with their long axes oriented perpendicularly to the plane of the layer. During motion the sheets flow freely over each other, but the molecules within each layer remain oriented and do not move between layers. In all these cases the orientation of the molecule can be described by the director.

A Cosserat medium [3] corresponds to a choice of three independent directors under the additional condition

$$d_{ia} d_{ib} = q_{ab} = const, \qquad a, b = 1, 2, 3. \tag{20}$$

This condition means that during the deformation process directors can only rotate as the rigid body. The motion of a Cosserat continuum is determined both by the displacement vector

$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t) \tag{21}$$

and the rotation vector

$$\varphi = \varphi(\mathbf{X}, t) \tag{22}$$

independent on it.

The deformation of a Cosserat continuum is described by the deformation gradients

$$\frac{\partial u_k}{\partial X_K}, \quad k=1,2,3, \quad K=1,2,3 \tag{23}$$

and

$$\frac{\partial \varphi_k}{\partial X_K}, \quad k=1,2,3, \quad K=1,2,3. \tag{24}$$

A generalized understanding of media with microstructure can be achieved using the mathematical structure of a fiber bundle (see e.g. [18, 24]).

A structure of differentiable fiber bundle is a six-tuple

$$(E, B, F, G, \pi, \psi), \tag{25}$$

where

the differentiable manifold E is the total space,
the differentiable manifold B is the base,
the differentiable manifold F is the fiber,
the Lie group G is the structural group,
the differentiable map $\pi : E \longrightarrow B$ is the projection,
ψ is a family of diffeomorphisms.

The following axioms must be satisfied.

(i) The local triviality:

$$\forall x \in B \;\; \exists U_\alpha \;\; \exists \psi_\alpha \in \psi : \psi_\alpha : U_\alpha \times F \longrightarrow \pi^{-1}(U_\alpha) \tag{26}$$

with

$$\pi \circ \psi_\alpha(x, f) = x \;\; for \;\; (x, f) \in U_\alpha \times F, \tag{27}$$

where U_α is an open covering of B.

(ii) If $x \in U_\alpha \cap U_\beta$, the diffeomorphism

$$\psi_{\beta x}^{-1} \circ \psi_{\alpha x} : F \longrightarrow F \tag{28}$$

coincides with the operation of an element $g \in G$.

For trivial (product) fiber bundle

$$E = B \times F. \tag{29}$$

Important special cases of fiber bundles are the tangent bundle, the vector bundle, the principal bundle, and the associated bundle [18, 24].

A vector bundle is a differentiable fiber bundle in which the fiber F is a linear space and the maps

$$\psi_{\alpha x} : F \longrightarrow F_x \tag{30}$$

are linear isomorphisms.

In a case of tangent bundle $T(M)$ of a differentiable manifold M,

$$E \equiv T(M) = \bigcup_{x \in M} T_x(M), \tag{31}$$

the base space is the manifold M, the fiber at any point $x \in M$ is the tangent space $T_x(M)$ (all the fibers are copies of the vector space $\mathbb{R}^n$). The projection is defined by

$$\pi : T(M) \longrightarrow M. \tag{32}$$

The structural group is $G = GL(n, \mathbb{R})$, i.e. the full linear group on the fiber $F = \mathbb{R}^n$.

The principal fiber bundle looks like a collection of copies of the structural group G sitting over the base manifold B. For each $x \in B$ the fiber $\pi^{-1}(x)$ is diffeomorphic with the structural group G. For example, the canonical trivial principal bundle is defined as the product

$$E = B \times G. \tag{33}$$

Below we present various fiber bundles which are used in the literature for describing the continua with microstructure.

The extra degrees of freedom in liquid crystals may be summarized by choosing $E = \mathbb{R}^3 \times \mathcal{R}$, where $\mathcal{R}$ is some manifold. The body should consist of points in physical space together with rod variables (a representative of $\mathcal{R}$) attached to each point. For example [16],

$$E = \mathbb{R}^3 \times \mathbb{P}^2 \tag{34}$$

corresponds to nematic liquid crystals with inextensible undirected rods. Here $\mathbb{P}^2$ is a real projective two–space (the unit two–sphere S^2 in $\mathbb{R}^3$ with antipodal points identified). It is used to model inextensible rods that have indistinguishing ends.

For cholesteric liquid crystals with inextensible directed rods the containing space is represented by

$$E = \mathbb{R}^3 \times S^2. \tag{35}$$

The two–sphere S^2 is used to model vectors that are free to point in any directions, but which are inextensible.

At least, we have

$$E = \mathbb{R}^3 \times \mathbb{R}^3 \tag{36}$$

for a general case of both extensible and directed rods.

The configuration space for the micromorphic continuum has the form [19]

$$E = \mathbb{R}^3 \times GL^+(3) \tag{37}$$

and that for the Cosserat continuum is written as

$$E = \mathbb{R}^3 \times SO(3), \tag{38}$$

where $GL^+(3)$ is the general linear group the elements of which have positive determinants, $SO(3)$ is the special orthogonal group.

The following fiber bundles

$$E = \mathbb{R}^3 \times (T(3) \triangleright SO(3)) \tag{39}$$

and

$$E = \mathbb{R}^3 \times (T(3) \triangleright SO(3)) \otimes \left\{ \underbrace{SO(3) \otimes \ldots \otimes SO(3)}_{N} \right\} \tag{40}$$

are utilized in the gauge theories of Cosserat continuum [1, 5, 13] (see also [2]) and continuum with N directors [15], respectively. Here $\triangleright$ is the semidirect product and $\left\{ \underbrace{SO(3) \otimes \ldots \otimes SO(3)}_{N} \right\}$ denotes the direct product repeated N times (the number of directors describing the microstructure).

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