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# Some remarks on Riemann's lecture

Petr Rys

To start I will talk about some historical facts about Riemann. He was born on 17-th of September in 1826. In 1846 he went to study to Berlin where he was taught by Jacobi, Dirichlet and Eisenstein. In 1849 he returned to Göttingen where his work was led by Gauß.

He was mostly concerned in physical problems, even though he wrote at the same time an excellent dissertation „The fundamentals of common theory of a complex variable". In this article he established a conception of Riemann's surface and its bounded function. At that time he also fell ill. These facts were taking away Riemann's attention from his work on the lecture leading to a habilitation procedure which he finished with no problems in a very short time. According to the common rules at German universities at that time he was obliged to make so called experimental lecture (Probevorlesung) in front of professors' advisory board. He could have chosen one from three of his suggested themes. He suggested these following themes:

1. History of problem of function development in trigonometric series.

2. Solving system of two quadratic equations with two unknowns.

3. About hypotheses lying in the fundamentals of geometry.

Riemann wrote a letter to his brother and he complained that he was prepared for the first two themes, but Gauß has chosen the third one. So he must start over again. However, this work was put away for many times because, as he wrote to his brother, he was too much concerned in physical problems. He began preparing this lecture two weeks after Easter in 1854 and five or six weeks later he finished it. The lecture was set up on 10-th of July in 1854. The professors of philosophy probably did not much understand this lecture but Gauß was very excited of the depth of Riemann's thoughts. Because he gave this lecture mostly in front of nonmathematical public, he formulated it just verbally, only sometimes he used a formula saying that relevant calculation may be easily deduced.

Really after publishing this lecture many works developing these themes appeared.

Problems introduced in this work we may divide into problems in:

1. topology
2. differential geometry
3. elementary geometry
4. physics

To the first point. Riemann establishes a term of topological space which he calls ,,ausgedehnte Mannigfaltigkeit". He does not mention an accurate definition of this term but he distinguishes discrete space that is more common and continuous space. In this continuous space he gives examples like our space (Raum) and space of colors and the others appear in higher mathematics. He introduces these spaces thanks to a passage from one point to another ,,infinitesimal small change of a certain element of this space" but about this problem we may speculate without metric terms. He sets a goal to make a term of $n$-dimensional topological space where he gives one dimensional continuous topological space at first and recurrently $n$-dimensional. He makes quantitative characteristics of a certain space by means of co-ordinate functions. He also talks about that the space does not have to be only $n$-dimensional, but it can have countable or also uncountable dimension. As examples he gives a space of functions given on certain interval or shapes of cubic figures. In the conclusion he says that it is possible to imagine that our space (Raum) is in fact discrete topological space.

To the second range of problems it means to differential geometry. It is necessary to bring metric into $n$-dimensional space in contradistinction to Euclidean geometry where the metric is established by means of distances of two points. Riemann introduces the metric so that each curve has its lengths and so that each curve is measurable with any other curve. As a curve he means points which comply with parametric equations $x = (x_1(t), \ldots, x_n(t))$. He is interested only in the functions $x_i(t)$ which differentials of degree one are continuous. So for the arc of a curve must hold that $ds = f(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$. He assumes that it is homogeneous function of degree one $f(x_1, \ldots, x_n, kdx_1, \ldots, kdx_n) = |k| f(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ which has a minimum in point $(x_1, \ldots, x_n)$. He takes these presumptions a priori, evidently according to Gauß's ,,Disquisitiones generales circa superficies curvas". A differential of the arc

$$ds = \sqrt{\sum_i \sum_j g_{ij} dx_i dx_j}$$

which is deduced here on the basis of Euclidean geometry for infinitesimally close points. A selection of this function could be various, for example $\sqrt[4]{dx_1^4 + \cdots + dx_n^4}$. Riemann knowns about these possibilities but in the following he considers only a case that $ds = \sqrt{\sum_i \sum_j g_{ij} dx_i dx_j}$. As he say the research of such spaces would be time-consuming and he would not give many new ideas of the science about space (Raum) and what more the results can not be expressed geometrically. Research of the common kind was made by Fisher who wanted to exists geodesics which means curves holding $\delta \int ds = 0$.

The easiest case of a space is when it is possible to transform the quadratic form defining its metric by linear transformation to the form $ds^2 = dy_1^2 + \cdots + dy_n^2$, which is called plane.

He establishes geodesic co-ordinates in which the geodesics have a form $y_i = a_i t$ and in point $A$ is $t = 0$. In point $A$ it acquires $ds^2 = \sum dy_i^2$. With such establishment covariant derivative is nothing else than derivative to these co-ordinates. This was used in 20's and 30's by Veblen in geometry of paths.

If we make development of

$$ds^2 = \sum dy_i^2 + \omega$$

where

$$\omega = -\frac{1}{12} R_{ik,\alpha\beta}(y_i dy_k - y_k dy_i)(y_\alpha dy_\beta - y_\beta dy_\alpha) + \ldots$$

expressions of third and higher degrees.
In $A$ it acquires

$$R_{ik,\alpha\beta} = \left( \frac{\partial^2 g_{i\alpha}}{\partial y_k \partial y_\beta} + \frac{\partial^2 g_{k\beta}}{\partial y_i \partial y_\alpha} - \frac{\partial^2 g_{i\beta}}{\partial y_k \partial y_\alpha} - \frac{\partial^2 g_{k\alpha}}{\partial y_i \partial y_\beta} \right)$$

($R$ is Riemann – Christoffel's tensor). Riemann describes quantity $\omega$ as infinitely small quantity of fourth degree of variables $(y_1 dy_2 - y_2 dy_1)$, $(y_1 dy_3 - y_3 dy_1)$ etc. if we divide it by area of infinitely small triangle with vertexes with co-ordinates $(0, \ldots, 0)$, $(y_1, \ldots, y_n)$, $(dy_1, \ldots, dy_n)$ we get a value of deviation of surface from a plane in a certain point of a certain direction. If we multiply it by $-3$ we get $K$ – Gauß's curvature to which it is possible to give a meaning in surfaces that

$$K = k_1 k_2$$

and

$$K \int ds = \sum \alpha_i - \pi$$

where $k_1, k_2$ are principal curvatures and $\sum \alpha_i$ is the sum of inner angles in a triangle.

## References

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Petr Rys
Department of Mathematics
J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem, Czech Republic